

Adaptable Colouring of Graph Products

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Abstract

A colouring of the vertices of a graph (or hypergraph) G is *adapted* to a given colouring of the edges of G if no edge has the same colour as both (or all) its vertices. The *adaptable chromatic number* of G is the smallest integer k such that each edge-colouring of G by colours $1, 2, \dots, k$ admits an adapted vertex-colouring of G by the same colours $1, 2, \dots, k$. (The adaptable chromatic number is just one more than a previously investigated notion of *chromatic capacity*.) The adaptable chromatic number of a graph G is smaller than or equal to the ordinary chromatic number of G . While the ordinary chromatic number of all powers G^k of G remains the same as that of G , the adaptable chromatic number of G^k may increase with k . We conjecture that for all sufficiently large k the adaptable chromatic number of G^k equals the chromatic number of G . When G is complete, we prove this conjecture with $k \geq 4$, and offer additional evidence suggesting it may hold with $k \geq 2$. We also discuss other products and propose several open problems.

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1 Introduction

The notion of adaptable chromatic number seems to have naturally arisen in several unrelated contexts. Our own interest [13] was motivated by considerations in the area of matrix partitions of graphs, trigraph homomorphisms, and full constraint satisfaction problems [7, 9, 10]. Another motivation in the context of hypergraph list colourings, with a possible application in job scheduling, is described in [14]. The same concept is also proposed, for the special case of complete graphs (and under a different name), in [5, 4], where it arose as a natural generalization of split graphs. (In this sense, it is similar to the motivation from [7], where a generalization of split graphs is also proposed.) In [1], the concept arose from colouring problems in metric spaces. Finally, in [2, 3] it was motivated by generalizations in Ramsey theory [2, 3, 18], cf. also [11].

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Several of these sources were unaware of each other (and we have ourselves only recently discovered most), so many results were proved in multiple places. Moreover, the definitions not only differ by names, but are also slightly mismatched - while in some one computes the largest number of colours that ensure a ‘monochromatic edge’ must occur, in others one computes the smallest number of colours in which this can be avoided, resulting in a mismatch of one.

To be specific, we define a vertex colouring of a graph G to be *adapted* to an edge colouring of G (with the same set of colours), if no edge has the same colour as both its vertices. (Neither the edge nor the vertex colouring have to be proper.) This generalizes in the obvious way when G is a multigraph or a hypergraph. The *adaptable chromatic number* of G , denoted $\chi_{ad}(G)$, is the smallest integer k such that every edge colouring of G with colours $1, 2, \dots, k$, admits an adapted vertex colouring with the same colours. The *adaptable choice number* $\text{ch}_{ad}(G)$ is the smallest integer k , such that for any edge colouring F of G and any lists $L(v), v \in V(G)$, with $|L(v)| \geq k, v \in V(G)$, there exists a list colouring of G adapted to F . (This means the colour of each vertex v is from the list $L(v)$.)

Note that if G is a subgraph of G' , we have $\chi_{ad}(G) \leq \chi_{ad}(G')$. It also follows from these definitions that $\chi_{ad}(G) \leq \chi(G)$. There exist graphs (uniform hypergraphs of any edge size) G with $\chi_{ad}(G) = \chi(G)$ and $\chi(G)$ arbitrarily high [11]. For graphs G , [13] shows that we can additionally require that G has arbitrarily high girth. (This solves Problem 1 from [11].) This is generalized to uniform hypergraphs (with any edge size) in [14], where one obtains hypergraphs G with arbitrarily high girth and arbitrarily high $\text{ch}(G) = \chi_{ad}(G) = \chi(G)$. Graphs G with $\chi_{ad}(G) = 2$ have been characterized in [11, 13], and with $\text{ch}_{ad}(G) = 2$ in [14]. Graphs with $\chi_{ad}(G) = t$ for $t > 2$ are NP-complete to recognize [13]. The adaptable chromatic and choice numbers of complete graphs have attracted much attention [5, 1, 2, 3, 4, 14, 13], since they tend to be far from their chromatic and choice numbers. The best current lower and upper bounds for $\chi_{ad}(K_n)$ are both of the order of \sqrt{n} [5]. For $\text{ch}_{ad}(K_n)$, the results are similar [14]. Concrete values of $\chi_{ad}(K_n)$ for small n are computed in [13]. For a general graph (multigraph) G with n vertices, the order of $\chi_{ad}(G)$ is at least $\chi(G)/\sqrt{n \log \chi(G)}$ [11] and at most $\sqrt{\Delta(G)}$ [3, 13, 14]. Adaptable choice numbers of graphs on surfaces and for other graph classes have been investigated in [6, 16]. Let $\text{mad}(G) = \max_{H \subseteq G} 2|E(H)|/|V(H)|$ be the *maximum average degree* of G . It is proved in [16] that for any graph G , $\text{ch}_{ad}(G) \leq \lceil \text{mad}(G)/2 \rceil + 1$. This implies that the adaptable choice number of any planar graph is at most four, while it is shown in [13] that there exist planar graphs with adaptable chromatic number equal to four. It also provides a proof of the fact that the adaptable chromatic number of a planar graph is at most four without using the four-colour theorem, answering a question from [13].

There are graphs G for which $\chi_{ad}(G) = \chi(G)$, and there are also graphs G for which $\chi_{ad}(G)$ and $\chi(G)$ are far apart [13]. We consider some constructions that maintain the chromatic number but increase the adaptable chromatic number. Even if $\chi_{ad}(G)$ and $\chi(G)$ are far apart originally, we may end up with $\chi_{ad}(G') = \chi(G')$ after a few steps in the construction. Take the following example construction. Let $G[n]$ be the graph obtained from G by replacing each vertex v with an independent set I_v of size n , joining a vertex in I_v to a vertex in I_w if and only if v is adjacent to w . Note that $\chi(G[n]) = \chi(G)$ for all integers n . Also note that $\chi_{ad}(G[n]) \leq \chi_{ad}(G[n'])$ if $n \leq n'$, since $G[n]$ is a subgraph of $G[n']$.

Theorem 1 *For any graph G , there is an integer $n(G)$ such that if $n \geq n(G)$, then $\chi_{ad}(G[n]) = \chi(G[n]) = \chi(G)$.*

Proof. Assume $\chi(G) = \chi(G[n]) = k$. It suffices to show that if n is sufficiently large, then there is a $(k-1)$ -edge colouring F of $G[n]$ such that any $(k-1)$ -vertex colouring of $G[n]$ is not adapted to F . The edge colouring F of $G[n]$ is defined as follows: randomly colour the edges of $G[n]$ by $k-1$ colours. It is well-known [5] that (when n is large) with high probability the following holds:

(*) For any edge vv' of G , for any colour i , if $X \subseteq I_v$ and $Y \subseteq I_{v'}$ and $|X|, |Y| \geq n/(k-1)$ then there is an edge between X and Y are coloured with colour i .

So there is an edge colouring F of $G[n]$ for which (*) holds. Assume c is a $(k-1)$ -vertex colouring of $G[n]$. We shall show that c is not adapted to F . For each vertex v of G , let $\phi(v)$ be a colour i such that $|c^{-1}(i) \cap I_v| \geq n/(k-1)$ (by pigeonhole principle, such a colour exists). Since $\chi(G) = k$, there is an edge vv' of G such that $\phi(v) = \phi(v') = i$. Let $X = c^{-1}(i) \cap I_v$ and $Y = c^{-1}(i) \cap I_{v'}$. Then $|X|, |Y| \geq n/(k-1)$. Hence there is an edge of colour i connecting a vertex of X and a vertex of Y . So c is not adapted to F . ■

2 Categorical power of graphs

The graph $G[n]$ is the *lexicographic product* [15] of G and \overline{K}_n . As we have seen, this product does not change the chromatic number of the graph, but, as n becomes larger, the adaptable chromatic number may increase until it reaches the chromatic number of G .

One naturally asks about other graph products. The *categorical product* $G \times H$ of two graphs G and H has vertex set $V(G) \times V(H)$ and in which $(x, y) \sim (x', y')$ if and only if $xx' \in E(G)$ and $yy' \in E(H)$.

The graph $G \times K_n$ can be obtained from the graph $G[n]$ by removing, for each edge uv of G , a perfect matching from the complete bipartite subgraph induced by $I_u \cup I_v$. By a proof analogous to that of Theorem 1, we can show the following fact.

Theorem 2 *For any graph G , there is an integer $n(G)$ such that if $n \geq n(G)$, then $\chi_{ad}(G \times K_n) = \chi(G \times K_n) = \chi(G)$.*

As a consequence of Theorem 2, if H is a graph with sufficiently large clique number $\omega(H)$, then $\chi_{ad}(G \times H) = \chi(G \times H) = \chi(G)$. Are there graphs H with $\omega(H) = \omega(G)$ for which we also have $\chi(G \times H) = \chi(G)$ and $\chi_{ad}(G \times H) = \chi(G)$? A natural candidate is that H be the product of copies of G . The *categorical power* G^n of G is recursively defined a $G^1 = G, G^{n+1} = G^n \times G$. It is well-known and easy to see that G^n has the same clique number and the same chromatic number as G . If $n \leq n'$ then G^n is a subgraph of $G^{n'}$, so we again have $\chi_{ad}(G^n) \leq \chi_{ad}(G^{n'})$. We conjecture that if n is large enough, then $\chi_{ad}(G^n)$ also reaches $\chi(G)$.

Conjecture 1 *For any graph G , there is an integer $n(G)$ such that if $n \geq n(G)$, then $\chi_{ad}(G^n) = \chi(G^n) = \chi(G)$.*

In this section we prove that the conjecture holds if G is a complete graph; in this case we show $n(G) \leq 4$.

The *resistance* $\text{res}(G)$ of a graph G is defined as the minimum number of edges of G whose removal reduces the chromatic number.

Theorem 3 *Suppose G is an n -vertex k -chromatic graph. If the resistance of G is at least $(k-1)^2(n \ln 2 + \ln(k-1))$, then $\chi_{ad}(G) = k$. In particular, if $n \geq \ln(k-1)/(1 - \ln 2)$ and the resistance of G is at least $(k-1)^2n$, then $\chi_{ad}(G) = k$.*

Proof. Let $F : E(G) \rightarrow \{1, 2, \dots, k-1\}$ be a random edge colouring of G , where each edge is coloured independently and each colour is chosen with equal probability. Let $s = (k-1)(n \ln 2 + \ln(k-1))$

Property \star : For any subset X of $V(G)$, if $|E[X]| \geq s$, then for any colour $j \in \{1, 2, \dots, k-1\}$, there is an edge $e \in E[X]$ with $F(e) = j$.

Claim With positive probability F has Property (\star) .

Let X be a subset of $V(G)$ with $|E[X]| \geq s$. For any colour j , the probability that no edge $e \in E[X]$ is coloured with colour j is at most $(1 - 1/(k-1))^s \leq e^{-s/(k-1)}$. The probability that there is a colour j such that no edge $e \in E[X]$ is coloured with colour j is at most $(k-1)e^{-s/(k-1)}$. The probability that there is a set X with $|E[X]| \geq s$ and a colour j such that no edge of $E[X]$ is coloured with colour j is strictly less than $2^n(k-1)e^{-s/(k-1)} \leq 1$. Therefore with positive probability F has Property \star .

It follows from this claim that there is a $(k-1)$ -edge colouring F of G which has Property \star . Now we show that for this edge colouring F of G , there is no $(k-1)$ -vertex colouring of G which is adapted to F . Assume to the contrary that there is an $(k-1)$ -vertex colouring c of G which is adapted to F . Let V_1, V_2, \dots, V_{k-1} be the $k-1$ colour classes (where $V_i = c^{-1}(i)$). Consider the union $E' = E[V_1] \cup E[V_2] \cup \dots \cup E[V_{k-1}]$. By our assumption, $|E'| \geq (k-1)s$. Hence there is an index i such that $|E[V_i]| \geq s$. By the claim above, there is an edge $e \in E[V_i]$ with $F(e) = i$. This is contrary to the assumption that c is adapted to F .

Observe that if $n \geq \ln(k-1)/(1 - \ln 2)$, then $(k-1)^2 n \geq (k-1)^2(n \ln 2 + \ln(k-1))$. Hence if the resistance of G is at least $(k-1)^2 n$, then $\chi_{ad}(G) = k$. \blacksquare

The following theorem shows that Conjecture 1 holds for complete graphs.

Theorem 4 For any positive integer n , $\chi_{ad}(K_n^4) = n$.

Proof. By Theorem 3, it suffices to show that K_n^4 have resistance at least $(n-1)^2(n^4 \ln 2 + \ln(n-1))$. Assume H is an $(n-1)$ -chromatic subgraph of K_n^4 . Then each n -clique of K_n^4 has at least one edge not contained in $E(H)$. Since K_n^4 is edge transitive, each edge of K_n^4 is contained in the same number of copies of n -cliques. Assume K_n^4 has m n -cliques and each edge of K_n^4 is contained in s n -cliques. Then $m|E(K_n^4)| = mn(n-1)/2 = s|E(K_n^4)| = sn^4(n-1)^4/2$. So $m = sn^3(n-1)^3$. Since each of the m n -cliques contains at least one edge in $E(G) \setminus E(H)$, and each edge of $E(G) \setminus E(H)$ is contained in s n -cliques in K_n^4 , we conclude that $E(G) \setminus E(H)$ contains at least $n^3(n-1)^3$ edges. So the resistance of G is at least $n^3(n-1)^3 \geq (n-1)^2(n^4 \ln 2 + \ln(n-1))$ (for $n \geq 4$). \blacksquare

3 Squares of K_n

Theorem 4 shows that K_n^4 has its adaptable chromatic number equal to its chromatic number. It is not known if the exponent 4 can be reduced. We do not know any n for which $\chi_{ad}(K_n^2) < n$. In this section, we prove that if there is a projective plane of order n or $n+1$, then $\chi_{ad}(K_n^2) = n$; and if there is a projective plane of order $n-1$, then $\chi_{ad}(K_n^3) = n$.

For a graph G , the *independence number* $\alpha(G)$ of G is the cardinality of a maximum independent set of G . The *independence ratio* $i(G)$ of G is defined as $i(G) = \alpha(G)/|V(G)|$. The following lemma is trivial and the proof is omitted.

Lemma 1 Suppose G has a k -edge colouring F such that for each colour i , the subgraph $G_i = (V(G), F^{-1}(i))$ has independence ratio less than $1/k$, then $\chi_{ad}(G) \geq k+1$.

Theorem 5 *If there is a projective plane of order n , then $\chi_{ad}(K_n^2) = n$.*

Proof. Since $\chi(K_n^2) = n$, it suffices to prove that $\chi_{ad}(K_n^2) \geq n$.

The vertex set of K_n is $\{1, 2, \dots, n\}$ and the vertex of K_n^2 is $\{(i, j) : 1 \leq i, j \leq n\}$. Assume there is a projective plane of order n . Then there are $n-1$ mutually orthogonal latin squares A_1, A_2, \dots, A_{n-1} . For each latin square A_k , for each $s \in \{1, 2, \dots, n\}$, Let $X_{k,s} = \{(i, j) : A_k(i, j) = s\}$. Since A_k is a latin square, $A_k(i, j) = s = A_k(i', j')$ implies that $i \neq i'$ and $j \neq j'$. Therefore each $X_{k,s}$ induces an n -clique in K_n^2 . Let X_k be the spanning subgraph of K_n^2 which is the vertex disjoint union of $\{X_{k,s} : s = 1, 2, \dots, n\}$. Then X_k is the vertex disjoint union of n copies of K_n , and hence has independence ratio $1/n$.

Next we show that if $k \neq k'$, then X_k and $X_{k'}$ are edge disjoint. Assume this is not true. Then there are integers s, s' such that $X_{k,s}$ and $X_{k',s'}$ have a common edge $(i, j) \sim (i', j')$. By definition, $A_k(i, j) = A_k(i', j') = s$ and $A_{k'}(i, j) = A_{k'}(i', j') = s'$. So $(A_k(i, j), A_{k'}(i, j)) = (A_k(i', j'), A_{k'}(i', j'))$, contrary to the assumption that A_k and $A_{k'}$ are orthogonal.

Colour the edges in X_k by colour k . We obtain an $(n-1)$ -edge colouring F of K_n^2 such that for each colour k , the subgraph $G_k = (V(G), F^{-1}(k))$ has independence ratio $1/n < 1/(n-1)$. By Lemma 1, we have $\chi_{ad}(K_n^2) \geq n$. \blacksquare

Theorem 6 *If there is a projective plane of order $n+1$, then $\chi_{ad}(K_n^2) = n$.*

Proof. Again it suffices to prove that $\chi_{ad}(K_n^2) \geq n$. Let A_1, A_2, \dots, A_{n-1} be $n-1$ mutually orthogonal latin squares of order $n+1$. Let A'_k be obtained from A_k by deleting the last row and last column. Observe that the element $A_k(n+1, n+1)$ occurs n times in A'_k . If $1 \leq s \leq n+1$, $s \neq A_k(n+1, n+1)$, then s occurs $(n-1)$ times in A'_k . Let $X_{k,s} = \{(i, j) : A'_k(i, j) = s\}$. If $s \neq A_k(n+1, n+1)$, then $X_{k,s}$ induces an $(n-1)$ -clique in K_n^2 . If $s = A_k(n+1, n+1)$, then $X_{k,s}$ induces an n -clique in K_n^2 . Let X_k be the spanning subgraph of K_n^2 which is the vertex disjoint union of $\{X_{k,s} : s = 1, 2, \dots, n+1\}$. Then X_k has independence ratio $(n+1)/n^2 < 1/(n-1)$.

As in the proof of Theorem 5, if $k \neq k'$, then since $A_k, A_{k'}$ are orthogonal, we have that X_k and $X_{k'}$ are edge disjoint. Colour the edges in X_k by colour k . We obtain an $(n-1)$ -edge colouring F of a subgraph of K_n^2 (some edges may remain uncoloured) such that for each colour k , the subgraph $G_k = (V(G), F^{-1}(k))$ has independence ratio less than $1/(n-1)$. By Lemma 1, we have again $\chi_{ad}(K_n^2) \geq n$. \blacksquare

Lemma 2 *For any positive integers n, k , $\chi_{ad}(K_{n+1}^{k+1}) \geq \chi_{ad}(K_n^k) + 1$.*

Proof. The vertex set of K_n is $V(K_n) = Z_n$ (the additions of the vertices of K_n will be in Z_n , i.e., modulo n).

First observe that for any positive integer k , K_n^{k+1} contains n vertex disjoint copies of K_n^k induced by $Y_i = \{(i + j_1, j_1, j_2, \dots, j_k) : j_1, j_2, \dots, j_k \in Z_n\}$ ($i = 0, 1, \dots, n-1$).

The vertex set of K_{n+1} is Z_{n+1} . Consider the vertex $v^* = (n, n, \dots, n)$ of K_{n+1}^{k+1} . The subgraph of K_{n+1}^{k+1} induced by the neighbourhood $N_{K_{n+1}^{k+1}}(v^*)$ of v^* is a copy of K_n^{k+1} . By the observation above, it contains n vertex disjoint copies of K_n^k . We denote these n copies of K_n^k by Y_0, Y_1, \dots, Y_{n-1} .

Assume $\chi_{ad}(K_n^k) = m$. Then there is an $(m-1)$ -edge colouring F of K_n^k (using colours $0, 1, \dots, m-2$) such that K_n^k has no $(m-1)$ -vertex colouring adapted to F . Since $\chi(K_n^k) = n$, we know that $m \leq n$. Let F' be an m -edge colouring of K_{n+1}^{k+1} defined as follows:

For $j = 0, 1, \dots, m-1$, the edges connecting v^* and Y_j are coloured by colour j . (Since $m \leq n$, Y_j is defined for $j = 0, 1, \dots, m-1$). The edges in Y_j , as a copy of K_n^k , are coloured by F (recall that F is an $(m-1)$ -edge colouring of K_n^k), except that when $j < m-1$, colour j is changed to colour $m-1$. The other edges are coloured arbitrarily. Now we shall show that K_{n+1}^{k+1} has no m -vertex colouring adapted to F' .

Assume c is an m -vertex colouring of K_{n+1}^{k+1} adapted to F' . Assume $c(v^*) = j$ for some $j \in \{0, 1, \dots, m-1\}$. Then no vertex in Y_j is coloured by colour j . Thus the restriction of c to Y_j is an $(m-1)$ -vertex colouring of Y_j adapted to the restriction of F' to Y_j . But this is contrary to our choice of F . ■

Corollary 1 *If there is a projective plane of order $n-1$, then $\chi_{ad}(K_n^3) = n$.*

4 Decomposition of K_n^2 into edge disjoint copies of K_n

In the previous section, we have shown that if there is a projective plane of order n , then K_n^2 can be edge partitioned into $n-1$ subgraphs X_k ($k = 1, 2, \dots, n-1$), and each X_k is a vertex disjoint union of n copies of K_n . So K_n^2 can be partitioned into $n(n-1)$ edge disjoint copies of K_n . The following theorem shows that the converse is also true.

Theorem 7 *There exist $n(n-1)$ edge-disjoint copies of K_n in K_n^2 if and only if there exists a projective plane of order n .*

Proof. Observe that K_n^2 can contain at most $n(n-1)$ edge-disjoint copies of K_n , since each copy of K_n in K_n^2 contains an edge of the form $(1, i)(2, j)$ and there are only $n(n-1)$ edges between vertices with first coordinate 1 and vertices with first coordinate 2.

Suppose that there exist $n(n-1)$ edge-disjoint copies of K_n in K_n^2 . Since this number coincides with the total number of edges $(1, i), (2, j)$ in K_n^2 , we may assume that the copies are numbered $K(i, j), i \neq j$ so that each $K(i, j)$ contains the edge $(1, i)(2, j)$. We let $L_s, s = 1, 2, \dots, n-2$, be the idempotent latin square (have i in position (i, i)) which has in position (i, j) with $i \neq j$ the unique v for which $(s+2, v)$ is a vertex of $K(i, j)$.

To show that L_s is indeed a latin square, we need to show that if $L_s(i, j) = L_s(i', j') = v$, then $i \neq i'$ and $j \neq j'$. Assume to the contrary that $i = i'$. Then the edge $((s+2, v), (1, i))$ is in both $K(i, j)$ and $K(i', j')$, contrary to the assumption that the copies of K_n are edge disjoint. Therefore L_s is indeed a latin square.

To show that these latin squares are orthogonal, we assume to the contrary that L_s and $L_{s'}$ are not orthogonal. Hence there are two indices (i, j) and (i', j') such that $L_s(i, j) = L_s(i', j') = v$ and $L_{s'}(i, j) = L_{s'}(i', j') = u$. Then $((s+2, v), (s'+2, u))$ is an edge of both $K(i, j)$ and $K(i', j')$, contrary to our assumption. So these idempotent latin squares are mutually orthogonal.

It is well known [19] that any set of $n-2$ (or even $n-3$) mutually orthogonal latin squares can be extended to $n-1$ mutually orthogonal latin squares, and hence imply the existence of a projective plane of order n . ■

5 Cartesian power of graphs

Another frequently studied graph product is the Cartesian product of graphs. The *Cartesian product* $G \square H$ of two graphs G and H has vertex set $V(G) \times V(H)$ and in which $(x, y) \sim (x', y')$ if and only if either $xx' \in E(G)$ and $y = y'$ or $x = x'$ and $yy' \in E(H)$. The *Cartesian power* $G^{\square n}$ of G is recursively defined a $G^{\square 1} = G, G^{\square n+1} = G^{\square n} \square G$. Again it is well-known and easy to see that $G^{\square n}$ has the same chromatic number as G . We also conjecture that if n is large enough, then $\chi_{ad}(G^{\square n})$ also increases to reach $\chi(G)$.

Conjecture 2 *For any graph G , there is an integer $m(G)$ such that if $n \geq m(G)$, then $\chi_{ad}(G^{\square n}) = \chi(G^{\square n}) = \chi(G)$.*

In this section, we prove that the conjecture is true for graphs G whose ultimate fractional chromatic number is strictly larger than $\chi(G) - 1$. In particular, the conjecture holds for circular complete graphs.

Lemma 3 *Suppose G is a graph, n is an integer and $i(G) < 1/(n-1)$. Then $\chi_{ad}(G^{\square(n-1)}) \geq n$. If moreover, we have $\chi(G) = n$, then $\chi_{ad}(G^{\square(n-1)}) = n$.*

Proof. Suppose $(x_1, x_2, \dots, x_{n-1})(y_1, y_2, \dots, y_{n-1})$ is an edge of $G^{\square(n-1)}$. By definition, there is an index $j \in \{1, 2, \dots, n-1\}$ such that $x_j \sim y_j$ and $x_i = y_i$ for $i \neq j$. Colour this edge by colour j . So we obtain an $(n-1)$ -colouring F of the edges of $G^{\square(n-1)}$. Consider the subgraph G_j induced by edges of colour j . For each sequence $(x_1, x_2, \dots, x_{j-1}, \star, x_{j+1}, \dots, x_{n-1})$, by replacing the symbol \star with vertices of G , we obtain a copy of G , denoted by $G_{(x_1, x_2, \dots, x_{j-1}, \star, x_{j+1}, \dots, x_{n-1})}$. So G_j is the vertex disjoint union of $|V(G)|^{n-2}$ copies of G . By our assumption, this graph G_j has independence ratio less than $1/(n-1)$. By Lemma 1, $\chi_{ad}(G^{\square(n-1)}) \geq n$. If moreover, $\chi(G) = n$, then $\chi(G^{\square(n-1)}) = n$, implying that $\chi_{ad}(G^{\square(n-1)}) \leq n$. Hence $\chi_{ad}(G^{\square(n-1)}) = n$. \blacksquare

Corollary 2 *For any integer n , $\chi_{ad}(K_n^{\square(n-1)}) = n$.*

In comparison to the categorical product, one may ask if there is a constant k such that for any n , $\chi_{ad}(K_n^{\square k}) = n$. The answer is no. Indeed, the maximum degree of $K_n^{\square k}$ is $k(n-1)$, and it is proved in [13] that for any graph G , $\chi_{ad}(G) \leq \lceil \sqrt{e(2\Delta(G) - 1)} \rceil$. So if $\sqrt{2ekn} \leq n-1$, then $\chi_{ad}(K_n^{\square k}) \leq n-1$.

For a graph G , the *ultimate fractional chromatic number* $\chi_F(G)$ is defined as $\chi_F(G) = \lim_{k \rightarrow \infty} \chi_f(G^{\square k})$. The *ultimate independence ratio* $I(G)$ of G is defined as $I(G) = \lim_{k \rightarrow \infty} i(G^{\square k})$. It is known [22] that for any G , $\chi_F(G)I(G) = 1$.

Theorem 8 *For any graph G , there is an integer m such that $\chi_{ad}(G^{\square m}) \geq \lceil \chi_F(G) \rceil$. If $\chi(G) < \chi_F(G) + 1$, then there is an integer m such that $\chi_{ad}(G^{\square m}) = \chi(G^{\square m}) = \chi(G)$.*

Proof. Assume $\lceil \chi_F(G) \rceil = n$. Then there is an integer k such that $i(G^{\square k}) < 1/(n-1)$. By Lemma 3, we have $\chi_{ad}(G^{\square kn}) \geq n$. \blacksquare

Suppose G is a graph and p, q are positive integers. A (p, q) -colouring of G is a mapping $f : V(G) \rightarrow \{0, 1, \dots, p-1\}$ such that for any edge uv of G , $q \leq |f(u) - f(v)| \leq p - q$. The *circular chromatic number* of a graph G is defined as

$$\chi_c(G) = \inf\{p/q : G \text{ has a } (p, q)\text{-colouring}\}.$$

It is known that for any graph G , $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ and $\chi_f(G) \leq \chi_F(G) \leq \chi_c(G)$ [22]. A graph is called *star-extremal* if $\chi_f(G) = \chi_c(G)$ [8]. Many circulant graphs were shown to be star-extremal in [8]. In particular, for any $p \geq 2q$, the *circular complete graph* $K_{p/q}$ with vertex set $\{0, 1, \dots, p-1\}$ and edge set $\{ij : q \leq |i-j| \leq p-q\}$ is star-extremal.

Corollary 3 *If G is star-extremal then $\chi_{ad}(G^{\square m}) = \chi(G^{\square m}) = \chi(G)$ for sufficiently large m . In particular, if $G = K_{p/q}$ is a circular clique, then $\chi_{ad}(K_{p/q}^{\square m}) = \chi(K_{p/q}) = \lceil p/q \rceil$ for sufficiently large m .*

6 Cartesian product of distinct graphs

For categorical product of two distinct graphs $G \times H$, we do not have any non-trivial upper or lower bounds on $\chi_{ad}(G \times H)$ in terms of $\chi_{ad}(G)$ and $\chi_{ad}(H)$. For Cartesian product, the situation is slightly better.

Theorem 9 *For any graphs G and H , $\chi_{ad}(G \square H) \leq \chi_{ad}(G)\chi_{ad}(H)$.*

Proof. Assume $\chi_{ad}(G) = k$ and $\chi_{ad}(H) = k'$. Let F be an edge colouring of $G \square H$, using colours $\{(i, j) : 1 \leq i \leq k, 1 \leq j \leq k'\}$. For each vertex x of H , let G_x be the subgraph of $G \square H$ induced by $V(G) \times \{x\}$. Let F_x be the edge colouring of G_x defined as $F_x(e) = i$ if $F(e) = (i, j)$ for some j . For each vertex y of G , the subgraph H_y and the edge colouring F_y are defined similarly.

For each x of H , let f_x be a k -colouring of G_x adapted to F_x . For each vertex y of G , let g_y be a k' -colouring of H_y adapted to F_y . Then $\phi(y, x) = (f_x(y), g_y(x))$ is a kk' -colouring of $G \square H$ adapted to F . ■

Theorem 10 *For any graphs G, H ,*

$$\chi_{ad}(G \square H) \leq ch_{ad}(G \square H) \leq \min\{ch_{ad}(G) + \lceil \text{mad}(H)/2 \rceil, ch_{ad}(H) + \lceil \text{mad}(G)/2 \rceil\}.$$

Proof. Assume $ch_{ad}(G) = k$ and $\lceil \text{mad}(H)/2 \rceil = k'$. It suffices to show that $ch_{ad}(G \square H) \leq k + k'$. Let F be an edge colouring of G . Let L be a list assignment which assigns to each vertex (y, x) of $G \square H$ a set $L(y, x)$ of $k + k'$ permissible colours. For each vertex y of G , H_y is a copy of H . By a result of Hakimi [12], H_y has an orientation in which each vertex (y, x) of H_y has outdegree at most k' . Let $L'_x(y)$ be obtained from $L(y, x)$ by removing all the colours of the outgoing edges of (y, x) in H_y . Then $L'_x(y)$ contains at least k permissible colours. For each vertex x of H , G_x is a copy of G has adaptable choosability k . So there is an L'_x colouring ϕ_x of G_x adapted to F . It is obvious that ϕ defined as $\phi(y, x) = \phi_x(y)$ is an L -colouring of $G \square H$. ■

7 Open questions

There are many papers on the relation between a colouring parameter of a product graph and that of its factor graphs. Let $f(n) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = n\}$. It is easy to see that for any graph G, H , $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. Hence $f(n) \leq n$. The well-known Hedetniemi's conjecture asserts that $f(n) = n$. However, whether or not $\lim_{n \rightarrow \infty} f(n) = \infty$ is also a longstanding open problem [17, 20, 21]. For adaptable chromatic number, the same question is also interesting. Let

$$\psi(n) = \min\{\chi_{ad}(G \times H) : \chi_{ad}(G) = \chi_{ad}(H) = n\}.$$

Question 1 *Is it true that $\lim_{n \rightarrow \infty} \psi(n) = \infty$?*

For chromatic number, it is obvious that $\max\{\chi(G \times H) : \chi(G) = \chi(H) = n\} = n$. However, for adaptable chromatic number, the situation is more complicated. Let

$$\Psi(n) = \max\{\chi_{ad}(G \times H) : \chi_{ad}(G) = \chi_{ad}(H) = n\}.$$

We do not know if the function Ψ is well-defined.

Question 2 *Is it true that for each n , there is a finite $\Psi(n)$?*

The functions $f(n)$ and $\psi(n)$ may have some connection. We believe that function $\psi(n)$ is related to the function $f(n)$ defined above. Let $g(n) = \min\{\chi_{ad}(G) : \chi(G) = n\}$. We expect the following question has a positive answer.

Question 3 *Is it true that $\lim_{n \rightarrow \infty} g(n) = \infty$?*

If $\lim_{n \rightarrow \infty} g(n) = \infty$, then there is a function $\varrho : \mathbb{N} \rightarrow \mathbb{N}$ such that $\chi(G) \leq \varrho(\chi_{ad}(G))$. So if $\min\{\chi_{ad}(G), \chi_{ad}(H)\} = n$, then

$$\chi_{ad}(G \times H) \leq \chi(G \times H) \leq \min\{\chi(G), \chi(H)\} \leq \varrho(n).$$

Thus Question 2 has a positive answer (indeed, even the function $\Psi'(n) = \max\{\chi_{ad}(G \times H) : \min\{\chi_{ad}(G), \chi_{ad}(H)\} = n\}$ is well-defined). Moreover, if $\lim_{n \rightarrow \infty} g(n) = \infty$, and one could prove $\lim_{n \rightarrow \infty} \psi(n) = \infty$, then we would also obtain $\lim_{n \rightarrow \infty} f(n) = \infty$.

We have proved (Theorem 9) that $\chi_{ad}(G \square H) \leq \chi_{ad}(G)\chi_{ad}(H)$. If $\chi_{ad}(G) = \chi_{ad}(H) = 2$, then $\chi(G), \chi(H) \leq 3$. Hence $\chi_{ad}(G \square H) \leq \chi(G \square H) \leq 3$. So in this case, the upper bound in Theorem 9 can be reduced by 1. We do not know if there are cases that this bound is sharp.

Question 4 *Are there graphs with $\chi_{ad}(G \square H) = \chi_{ad}(G)\chi_{ad}(H)$?*

We have proved that for any graph G , there is an integer m such that $\chi_{ad}(G^{\square m}) \geq \chi_f(G)$. This should also be true (and hopefully not too difficult to prove) for the categorical power.

For Conjecture 1, we do not really have much evidence that $\chi_{ad}(G^n) = \chi(G)$ (for sufficiently large n) beyond the case of complete graphs. On the other hand, we also do not know any graph G for which $\chi_{ad}(G^2) < \chi(G)$. In particular, for the complete graphs, we wonder whether the fourth power in Theorem 4 can be lowered, and, in particular, whether or not already the square of each K_n satisfies $\chi_{ad}(K_n^2) = n$.

Question 5 *Is there an integer n such that $\chi_{ad}(K_n^2) < n$?*

If such an n exists, then there is no projective plane of order n or $n + 1$. The smallest integer n for which there is no projective plane of order n or $n + 1$ is 14. The next two such integers are 20, 21.

References

- [1] A. Archer, *On the upper chromatic numbers of the reals*, Discrete Math. 214 (2000) 65–75.
- [2] G.R. Brightwell, Y. Kohayakawa, *Ramsey properties of orientations of graphs*, Random Structures and Algorithms 4 (1993) 413–428.
- [3] M. Cochand, P. Duchet, *A few remarks on orientations of graphs and Ramsey theory*, in **Irregularities of Partitions** (G. Halász, V.T. Sós eds.), Algorithms and Combinatorics 8 (1989) pp. 39–46.
- [4] M. Cochand, G. Károlyi, *On a graph coloring problem*, Discrete Math. 194 (1999) 249–252.
- [5] P. Erdős, A. Gyárfás, *Split and balanced colorings of complete graphs*, Discrete Math. 200 (1999) 79–86.
- [6] L. Esperet, M. Montassier, X. Zhu, *Adapted list colouring of planar graphs*, manuscript, 2007.
- [7] T. Feder and P. Hell, *Full constraint satisfaction problems*, SIAM J. Comput., **36** (2006) 230-246.
- [8] G. Gao and X. Zhu, *Star-extremal graphs and lexicographic product*, Discrete Mathematics, **152** (1996), 147-156.
- [9] T. Feder, P. Hell, S. Klein, and R. Motwani, *Complexity of list partitions*, SIAM J. Discrete Mathematics **16** (2003) 449-478.
- [10] T. Feder, P. Hell, , W. Xie, *Matrix partitions with finitely many obstructions*, Electron. J. Combin. 14 (2007), no. 1, RP 58, 17 pp.
- [11] J.E. Greene, *Chromatic capacities of graphs and hypergraphs*, Discrete Math. 281 (2004) 197–207.
- [12] S. L. Hakimi, *On the degree of the vertices of a directed graph*, J. Franklin Inst. 279(1965), 290-308. MR31 #4736.
- [13] P. Hell and X. Zhu, *Adaptable chromatic number of graphs*, European J. Combinatorics, 29 (2008) 912–921.
- [14] A. Kostochka and X. Zhu, *Adapted list coloring of graphs and hypergraphs*, SIAM J. Discrete Math., to appear.
- [15] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, John Wiley & Sons, New York, 2000.
- [16] M. Montassier, A. Raspaud and X. Zhu, *An upper bound on adaptable choosability of graphs*, European Journal of Combinatorics, doi:10.1016/j.ejc.2008.06.003
- [17] S. Poljak and V. Rödl, *On the arc-chromatic number of a digraph*, *J. Combin. Theory B* **31** (1981) 190–198.
- [18] V. Rödl, *A generalization of Ramsey theorem*, in **Graphs, Hypergraphs, and Block Systems**, Zielona Gora 1976, 211–220.
- [19] S.S. Shrikhande, *A note on mutually orthogonal latin squares*, *The Indian Journal of Statistics A* 23 (1961) 115–116.
- [20] C. Tardif and D. Wehlau, *Chromatic numbers of products of graphs: The directed and undirected versions of the Poljak-Rödl function*, Journal of Graph Theory 51 (1) (2006), 33-36.

- [21] X. Zhu, *A survey on Hedetniemi's conjecture*, Taiwanese Journal of Mathematics 2 (1998), 1-24.
- [22] X. Zhu, *On the bounds of ultimate independence ratios of graphs*, Discrete Mathematics, **156** (1996) 229–236.