

MACM 101-D3, Fall 2004, SFU Surrey

Midterm #2 Preparation.

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Welcome to the second MACM-101 midterm.

- The duration of the midterm is 80 minutes.
 - You are not allowed access to any materials with the exception of your crib-sheet.
 - For each of the exercises it is more important that you explain and justify your calculations, than the actual result.
 - In absence of justifications, a numeric result, even if correct, may be disconsidered entirely.
 - When you arrive at a formula that contains just numbers (e.g. $\frac{5! \cdot 20!}{4 \cdot 3^{11}}$), it is not necessary to calculate the final numeric result.
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1. Expand the following:

$$(a) \sum_{k=2}^6 k^2 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 4 + 9 + 16 + 25 + 36 = 90$$

$$(b) \sum_{k=0}^4 P(5, k) = \sum_{k=0}^4 \frac{5!}{(5-k)!} = 5! \cdot \sum_{k=0}^4 \frac{1}{(5-k)!} = 5! \cdot \left(\frac{1}{5!} + \frac{1}{4!} + \frac{1}{3!} + \frac{1}{2!} + \frac{1}{1!} \right) = 1 + 5 + 5 \cdot 4 + 5 \cdot 4 \cdot 3 + 5 \cdot 4 \cdot 3 \cdot 2 = 206$$

2. Expand and calculate the following:

$$(a) \prod_{k=2}^6 \frac{\binom{k}{k-2}}{\binom{k+1}{k-1}} = \frac{\binom{2}{0}}{\binom{3}{1}} \cdot \frac{\binom{3}{1}}{\binom{4}{2}} \cdot \frac{\binom{4}{2}}{\binom{5}{3}} \cdot \frac{\binom{5}{3}}{\binom{6}{4}} \cdot \frac{\binom{6}{4}}{\binom{7}{5}} = \frac{\binom{2}{0}}{\binom{7}{5}} = \frac{1}{\binom{7}{2}} = \frac{1}{\frac{7 \cdot 6}{2 \cdot 1}} = \frac{1}{21}$$

$$(b) \sum_{k=0}^n \frac{P(n, k)}{P(k, k)} = \sum_{k=0}^n \frac{n!}{(n-k)! k!} = \sum_{k=0}^n \frac{n!}{(n-k)! \cdot k!} = \sum_{k=0}^n \binom{n}{k} = 2^n$$

3. Consider A, B, C finite sets. Prove the following property (distributivity of Cartesian product over union):

$$(A \cup B) \times C = (A \times C) \cup (B \times C)$$

Proof:

$$\begin{aligned} (A \cup B) \times C &= \\ \text{(using the definition of cartesian product)} &= \{(x, y) \text{ pair} \mid x \in A \cup B \wedge y \in C\} = \\ \text{(using the definition of union)} &= \{(x, y) \text{ pair} \mid (x \in A \vee x \in B) \wedge y \in C\} = \\ \text{(distributivity of } \wedge \text{ over } \vee) &= \{(x, y) \text{ pair} \mid (x \in A \wedge y \in C) \vee (x \in B \wedge y \in C)\} = \\ \text{(using the definition of cartesian product)} &= \{(x, y) \text{ pair} \mid [(x, y) \in A \times C] \vee [(x, y) \in B \times C]\} = \\ \text{(using the definition of union)} &= A \times C \cup B \times C \end{aligned}$$

4. Show that $\forall n, k$, where $k, n \in \mathbb{N}^*$ and $k < n - 1$:

$$\binom{n}{k} - \binom{n}{k+2} = \binom{n+1}{k+1} - \binom{n+1}{k+2}$$

Proof:

$$\binom{n}{k} - \binom{n}{k+2} = \binom{n}{k} + \binom{n}{k+1} - \binom{n}{k+1} - \binom{n}{k+2} = \binom{n+1}{k+1} - \binom{n+1}{k+2}$$

5. A US zip code is made of 5 digits (e.g. 12345). A Canadian postal code is made of six symbols, alternating letters and digits (e.g. V5A 1S6). You are asked below to calculate how many such codes there are, based on increasing restrictions.

Some steps are based on steps before them. You can skip steps which you don't know.

- (a) How many possible US zip codes (as above) are there? (N_1)

Solution:

The choice of each digit is independent of the choice of each other digit.

Consider the set of digits $D = 0..9$, $|D| = 10$. The problem can be rephrased as the cardinality of the set D^5 . Using the rule of product:

$$N_1 = |D^5| = |D|^5 = 10^5 = 100,000$$

- (b) No¹ US zip code can start with '0'. How many US zip codes are left? (N_2)
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Solution:

The choice of each digit is independent of the choice of each other digit.

Consider the set of non-zero digits $D^* = 1..9$, $|D^*| = 9$. The problem can be rephrased as the cardinality of the set $D^* \times D^4$. Using the rule of product:

$$N_2 = |D^* \times D^4| = |D^*| \times |D|^4 = 9 \cdot 10^4 = 90,000$$

- (c) How many possible Canadian postal codes (as above) are there? (N_3)
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Solution:

The choice of each character is independent of the choice of each other character.

Consider the set of digits $D = 0..9$, $|D| = 10$ and the set of alphabetic letters $\mathcal{L} = \{A, B, C, \dots Z\}$, $|\mathcal{L}| = 26$. The problem can be rephrased as the cardinality of the set $\mathcal{L} \times D \times \mathcal{L} \times D \times \mathcal{L} \times D$. Using the rule of product:

$$N_3 = |\mathcal{L} \times D \times \mathcal{L} \times D \times \mathcal{L} \times D| = |\mathcal{L}| \times |D| \times |\mathcal{L}| \times |D| \times |\mathcal{L}| \times |D| = 26^3 \cdot 10^3 = 17,576,000$$

- (d) Canadian postal codes can start with only one of 13 letters indicating the province or territory (e.g. 'V' for British Columbia). How many Canadian postal codes are left? (N_4)
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Solution:

Consider the set of the 13 allowable first letters for Canadian postal codes \mathcal{L}_{13} , $|\mathcal{L}_{13}| = 13$. The problem can be rephrased as the cardinality of the set $\mathcal{L}_{13} \times D \times \mathcal{L} \times D \times \mathcal{L} \times D$. Using the rule of product:

$$N_4 = |\mathcal{L}_{13} \times D \times \mathcal{L} \times D \times \mathcal{L} \times D| = |\mathcal{L}_{13}| \times |D| \times |\mathcal{L}| \times |D| \times |\mathcal{L}| \times |D| = 13 \cdot 26^2 \cdot 10^3 = 8,788,000$$

- (e) How many strings in total are there that can represent US zip codes and Canadian postal codes (following all the above restrictions)? (N_5)

¹This is probably not true in real life.

Solution:

The set of possible US zip codes (calculated in (5b)) and the set of Canadian postal codes (calculated in (5d)) are disjoint (codes have different lengths, etc). The rule of sum can be applied:

$$N_5 = N_2 + N_4 = 90,000 + 8,788,000 = 8,878,000$$

6. “Crazy Mike’s” pizza place is making a new kind of pizza: the “Sausage Extreme”. Each such pizza is covered with exactly 16 slices of 3 types of sausage: “Italian”, “Turkey” and “Bologna”.

If you order 4 “Sausage Extreme” pizzas, and you find that each has more than 10 slices of sausage of the same type, you win a trip to Buffalo, New York.

Calculate the proportion of winning orders of 4 pizzas, compared to all possible orders of 4 pizzas. Explain.

Solution: Consider:

- Italian: I
- Turkey: T
- Bologna: B

Each pizza can be considered a multiset of size $r = 16$ with a base set $\{I, T, B\}$ of cardinality $n = 3$.

The total number (N_{16}^3) of different sausage distributions for one pizza is the number of combinations with repetitions of n taken by r .

$$N_{16}^3 = \binom{n+r-1}{r} = \binom{16+3-1}{16} = \binom{18}{16} = \binom{18}{2} = \frac{18 \cdot 17}{2} = 153$$

A pizza with more than 10 slices of sausage of the same type will have 11 or more such slices.

Given that there are 3 different types of sausage, each such distribution will be based on 11 slices of the same kind (3 possibilities: I, T, B) and the distribution of the remaining $16 - 11 = 5$ from the set (independent from the allocation of the first 11).

The number (N_5^3) of distributions of the last 5 slices is a problem of combinations with repetitions (similarly with above).

$$N_5^3 = \binom{5+3-1}{5} = \binom{7}{5} = \binom{7}{2} = \frac{7 \cdot 6}{2} = 21$$

4 pizzas can be considered a multi-list of size $R = 4$ with a base set: the set of all possible sausage distributions for one pizza: N_{16}^3 . The number

of such 4-pizza orders is combinations with repetitions of N_{16}^3 taken as R :

$$N_{153}^4 = \binom{153 + 4 - 1}{4} = \binom{156}{4} = 23,738,715$$

Similarly, each winning order of $R = 4$ pizzas is a multiset of size 4 from the base set of pizzas with more than 10 slices of the same kind of sausage ($3 \cdot N_5^3 = 63$ possibilities). The number of winning 4-pizza orders is combinations with repetitions of $3 \cdot N_5^3$ taken as R :

$$N_{63}^4 = \binom{63 + 4 - 1}{4} = \binom{66}{4} = 720,720$$

The proportion of winning pizza orders compared to all possible pizza orders is:

$$p = \frac{N_{63}^4}{N_{153}^4} = \frac{720720}{23738715} \approx 0.0304 = 3.04\%$$