SFU MACM-101-D3 2004-2 week 12

Manuel Zahariev

E-mail: manuelz@cs.sfu.ca

September 9, 2004

$Revision$: 1.10

Logic Statements

Definition 1 (Statement) *A declarative sentence which –when uttered– is either true or false, cannot be both true and false or neither true nor false is a statement (also called a proposition).*

Observation 1 *Statements have truth value: true or false.*

Logic Statements

Examples of statements:

- Glass is brittle.
- Water is liquid at room temperature.
- The price of oil is decreasing.
- Today is Tuesday.

Logic Statements

Examples of non-statements:

- You should go to school.
- You are fired!
- Great fries!
- What is the height of Mt. Everest?
- $x + 3 > 8$
- The King of France is bald.

Observation 2 *Utterances in natural language are inherently ambiguous. This justifies the use of a mathematical aparatus to deal with truth values of complex statements.*

Note 1 *Propositional logic is the systematic study of the truth values of statements.*

Combining Statements

Definition 2 (Primitive Statement) *A statement that cannot be broken into simpler statements is a primitive statement.*

Definition 3 (Compound Statements; Connectives) *Statements can be transformed through negation. Statements can be combined into compound statements using connectives.*

Negation and Connectives

Consider p and q logical statements.

- *Negation*: $\neg p$ read "Not p".
- *Conjunction*: $p \wedge q$ read "p and q"
- *Disjunction*: $p \vee q$ read "p or q"
- *Exclusive or*: $p \veebar q$ read "Exactly one of p and q is true"
- *Implication*: $p \rightarrow q$ read "p implies q"
- *Biconditional*: $p \leftrightarrow q$ read "p if and only if q"

The Definition of Negation

Consider p a logical statement.

p	$\neg p$
0	1
1	0

The Definitions of Connectives

Consider p and q logical statements.

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Definition 4 (Tautology) *A compound statement is called a* tautology *(noted T_0) if it is* true *for all truth values of its primitive statements.*

Definition 5 (Contradiction) *A compound statement is called a* contradiction *(noted F_0) if it is* false *for all truth values of its primitive statements.*

Logical Equivalence

Definition 6 *Two statements s_1 and s_2 are said to be* logically equivalent *(written $s_1 \Leftrightarrow s_2$) when the statement s_1 is true (respectively false) whenever the statement s_2 is true (respectively false).*

Observation 3 *Logically equivalent statements have identical truth tables.*

Observation 4 *For p and q logically equivalent statements $p \Leftrightarrow q$, the biconditional $p \leftrightarrow q$ is a tautology.*

Logical Equivalence: Example

p	$\neg p$	q	$p \rightarrow q$	$\neg p \vee q$
0	1	0	1	1
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1

$$p \rightarrow q \Leftrightarrow \neg p \vee q$$

Logical Equivalence: Example 2

p	q	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

$$p \leftrightarrow q \Leftrightarrow (p \to q) \land (q \to p)$$

The problem with truth tables

How many truth values are there in truth table for a compound statement with n primitive statements?

The problem with truth tables

How many truth values are there in truth table for a compound statement with n primitive statements?

Observation 6 *A truth table for a compound statement with n primitive statements has 2^n truth values (exponential in n).*

Writing truth tables

How do you choose the truth values for primitive statements in a truth table?

Writing truth tables

How do you choose the truth values for primitive statements in a truth table?

Observation 8 *Choose truth values for n primitive statements in a compound statement:*

1. *Write all numbers with n digits in base 2, in order.*
2. *For each such binary number, for each k from 1 to n, assign to the k-th primitive statement the value of the k-th digit in the binary number.*

The problem with truth tables

Observation 9 *Exponentials grow very, very quickly.*

- $2^1 = 2$
- $2^2 = 4$
- $2^3 = 8$
- $\ldots$
- $2^{10} = 1024$
- $\ldots$
- $2^{20} \approx 10^6$
- $\ldots$
- $2^{30} \approx 10^9$

The Laws of Logic

Observation 10 *All laws of logic (below) are defined for* primitive statements.

Law 1 (Double negation)

$$\neg p \Leftrightarrow p$$

Proof:

p	$\neg p$	$\neg\neg p$
0	1	0
1	0	1

The Laws of Logic

Law 2 (DeMorgan's)

$$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$$

$$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$$

Proof:

p	q	$\neg(p \vee q)$	$(\neg p \wedge \neg q)$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

The Laws of Logic

Law 3 (Commutative)

$$p \vee q \Leftrightarrow q \vee p$$

$$p \wedge q \Leftrightarrow q \wedge p$$

Law 4 (Associative)

$$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

Law 5 (Distributive)

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

The Laws of Logic

Law 6 (Idempotent)

$$p \vee p \Leftrightarrow p$$

$$p \wedge p \Leftrightarrow p$$

Law 7 (Identity)

$$p \vee F_0 \Leftrightarrow p$$

$$p \wedge T_0 \Leftrightarrow p$$

Law 8 (Inverse)

$$p \vee \neg p \Leftrightarrow T_0$$

$$p \wedge \neg p \Leftrightarrow F_0$$

The Laws of Logic

Law 9 (Domination)

$$p \vee T_0 \Leftrightarrow T_0$$

$$p \wedge F_0 \Leftrightarrow F_0$$

Law 10 (Absorption)

$$p \vee (p \wedge q) \Leftrightarrow p$$

$$p \wedge (p \vee q) \Leftrightarrow p$$

Logic Duality

Observation 11 *All laws of logic, with the exception of the first law (double negation)* come in pairs.

Definition 7 *Let s be a statement which contains no other connectives than $\wedge$ and $\vee$, the* dual *of s, written s^d is a statement where every $\wedge$ symbol in s is replaced with a $\vee$ and every $\vee$ is replaced with a $\wedge$.*

Example:

$$s = (\neg p \vee q) \wedge (\neg q \vee p)$$

$$s^d = (\neg p \wedge q) \vee (\neg q \wedge p)$$

Logic Duality

Theorem 1 (The Principle of Duality) *Let s and t be statements which contain no other connectives than $\wedge$ and $\vee$. If $s \Leftrightarrow t$ then $s^d \Leftrightarrow t^d$*

Proof : in Chapter 15 of the textbook (not required).

Observation 12 *For a statement s which contains no other connectives than $\wedge$ and $\vee$, $(s^d)^d \Leftrightarrow s$.*

Observation 13 *Any statement is equivalent to a statement which contains no other connectives than $\wedge$ and $\vee$.*

Substitution rules

Axiom 1 *For a compound statement P, which is a tautology, for a primitive statement p, if each occurrence of p is replaced with another statement q (not necessarily primitive) the resulting statement P_1 will also be a tautology.*

Axiom 2 *For a compound statement P, which is a tautology, for an arbitrary statement p that appears in P, if $q \Leftrightarrow p$, then for the statement P_1 obtained from P where each occurrence of p is replaced with q, $P \Leftrightarrow P_1$.*

Applications: programming

The SFU student registration system.

- If a student is registered for a course, they have paid to SFU more than the $100 deposit.
- Do you apply to a student a late course drop penalty?
- Does a student have the prerequisites for taking a given course?
- Does a student qualify for graduation?

Build a calendar application that displays the number of days in a month (including for leap years).

Logical implication

Definition 8 (Logical implication) *For p and q statements so that $p \to q$ is a tautology, $p \Rightarrow q$ (read p implies q)*

$$(p \Rightarrow q) \Leftrightarrow ((p \to q) \Leftrightarrow T_0)$$

Observation 14 *Logical equivalence is stronger than implication.*

Rules of inference

Rule 1 (Modus Ponens)

$$[p \wedge (p \to q)] \to q$$

Rule 2 (Law of the Syllogism)

$$[(p \to q) \wedge (q \to r)] \to (p \to r)$$

Rule 3 (Modus Tollens)

$$[(p \to q) \wedge \neg q] \to \neg p$$

Rules of inference

Rule 4 (Rule of Conjunction)

$$\begin{array}{r} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Rule 5 (Rule of Disjunctive Syllogism)

$$[(p \vee q) \wedge \neg p] \to q$$

Rule 6 (Rule of Contradiction)

$$(p \to F_0) \to \neg p$$

Rules of inference

Rule 7 (Rule of Conjunctive Simplification)

$$(p \wedge q) \rightarrow p$$

Rule 8 (Rule of Disjunctive Amplification)

$$p \rightarrow (p \vee q)$$

Rules of inference

Rule 9 (Rule of Conditional Proof)

$$[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)] \rightarrow r$$

Rule 10 (Rule of Conditional Proof)

$$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$$