SFU MACM-101-D3 2004-2 week 2

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Open statements

Note 1 *Open statements contain unspecified values (variables).*

Examples:

- x is greater than 30.
- $p \vee q$ is false.
- He studies logic.

Definition 1 *A declarative sentence is an open statement if:*

1. *it contains at least one variable.*
2. *it is not a statement.*
3. *it becomes a statement when variables are replaced by specific (allowable) values.*

Closing open statements

Observation 1 *Open statements are not statements. Closing open statements results in statements.*

Quantification: classes of values

Definition 2 *The totality of allowable choices for values of an open statement are called the universe of discourse for the open statement.*

Special places in the universe of the discourse:

- No values
- One value
- All possible values

Quantifiers:

- $\forall x$ for all x, also called *universally quantified*
- $\exists x$ for some x, also called *existentially quantified*

Quantification: mathematical examples

Some integers are even.

$$\exists x[x \text{ is even}]$$

All integers have positive or zero squares.

$$\forall x[x^2 \geq 0]$$

also:

$$\forall x[(x^2 > 0) \vee (x^2 = 0)]$$

Logical equivalence quantification

Definition 3 *Consider $p(x)$ and $q(x)$ open statements.*

p is called logically equivalent to q. if $\forall x[p(x) \leftrightarrow q(x)]$ is true.

Written: $p(x) \Leftrightarrow q(x)$)

Equivalence example

$p(x)$: x is an equilateral triangle

$q(x)$: x is a triangle with three 60^o angles

$$\forall x[p(x) \leftrightarrow q(x)]$$

Implication quantification

Definition 4 *Consider $p(x)$ and $q(x)$ open statements.*

p is said to imply *q if $\forall x[p(x) \rightarrow q(x)]$ is true.*

Written: $p(x) \Rightarrow q(x)$

Implication example

$p(x)$: x is a rational number

$q(x)$: x is a real number

$$\forall x[p(x) \rightarrow q(x)]$$

Quantification and implication

$$\forall x[p(x)] \Rightarrow \exists x[p(x)]$$

Related statements

Definition 5 *Given open statements $p(x)$ and $q(x)$, for the statement $\forall x[p(x) \rightarrow q(x)]$:*

- *the* counterpositive*: $\forall x[\neg q(x) \rightarrow \neg p(x)]$*
- *the* converse*: $\forall x[q(x) \rightarrow p(x)]$*
- *the* inverse*: $\forall x[\neg p(x) \rightarrow \neg q(x)]$*

Observation 2 *The counterpositive statement* is logically equivalent *to the original statement.*

Quantified statements in one variable

$$\exists x[p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x)] \wedge [\exists x\, q(x)] \quad (1)$$

$$\exists x[p(x) \vee q(x)] \Leftrightarrow [\exists x\, p(x)] \vee [\exists x\, q(x)] \quad (2)$$

$$\forall x[p(x) \wedge q(x)] \Leftrightarrow [\forall x\, p(x)] \wedge [\forall x\, q(x)] \quad (3)$$

$$[\forall x p(x)] \wedge [\forall x q(x)] \Rightarrow \forall x[p(x) \vee q(x)] \quad (4)$$

Negation of quantified statements

$$\neg[\forall x\, p(x)] \Leftrightarrow \exists x\, \neg p(x) \tag{5}$$
$$\neg[\exists x\, p(x)] \Leftrightarrow \forall x\, \neg p(x) \tag{6}$$

Combining quantifiers

$$\exists x \exists y\, p(x, y) \Leftrightarrow \exists y \exists x\, p(x, y) \tag{7}$$
$$\forall x \forall y\, p(x, y) \Leftrightarrow \forall y \forall x\, p(x, y) \tag{8}$$
$$\tag{9}$$

The following is generally not true:

$$\exists x \forall y\, p(x, y) \Leftrightarrow \forall y \exists x\, p(x, y)$$