SFU MACM-101-D3 2004-2 week 4

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Mathematical terminology

definition statements (propositions)

- definition : a *labelling* or *naming* of a given entity or class of entities.
- axiom (postulate) : an utterance that is said to be a true statement.
- theorem : a statement that can be proven true by reasoning from axioms.
- corrolary : a statement (theorem) that is a direct consequence of a notable theorem.
- conjecture : an utterance that is believed to be a true statement, but which hasn't been proven or disproven yet.

Mathematical terminology

For statements p and q:

- p is sufficient for q ($p \rightarrow q$)
- p is necessary for q ($q \rightarrow p$)
- p is necessary and sufficient q ($p \leftrightarrow q$)
- p if and only if q ($p \leftrightarrow q$)

Mathematical terminology

The form of definitions:

We call something L *if and only if* P.

Observation 1 *Axioms, theorems are of the form: P, where P is a statement.*

Observation 2 *Many propositions in mathematics are formulated as quantified open statements.*

Observation 3 *Many theorems are formulated as logical implications or as logical equivalences (conjunctions of opposed logical implications).*

Mathematical terminology

Observation 4 *The proof of theorems is the central activity in the study of mathematics.*

Observation 5 (The life cycle of mathematical propositions)

- *create a new Conjecture.*
- *Conjecture proven false.*
- *Conjecture* $\rightsquigarrow$ *theorem.*
- *Conjecture* $\rightsquigarrow$ *axiom.*

Observation 6 *The truth of axioms cannot be demonstrated, but is stated (or postulated).*

Applications of Mathematics

Observation 7 *The application of theorems in specific cases, where reality fits the premise of the theorem.*

Example: finding your location using a map and a compass.

Observation 8 (The rule of Universal Generalization) *For universally quantified open statements, the truth of the open statement with a substitution of the variable with any given value in the universe will follow from the truth of the universally quantified open statement.*

Mathematics

Observation 9 *The truth of a given universally quantified statement (conjecture) can be proven through exhaustive coverage of the universe (the method of* exhaustion*).*

Observation 10 *The falsity of a universally quantified statement (conjecture) can be proven as a side effect of the method of exhaustion.*

http://www-gap.dcs.st-and.ac.uk/ history/HistTopics/Fermat's_last_theorem.html

Reasoning using quantified statements

Observation 11 *Use of the method of exhaustion can be very demanding or even impossible, depending on the size of the universe.*

Observation 12 (The rule of Universal Generalization) *Propositional logic can be applied to instantiations of universally quantified statements to fixed, but arbitrary values.*

Example: Geometry

http://www.math.uncc.edu/˜droyster/courses/spring99/math3181/classnotes/axioms.pdf
(from Dr. David C. Royster, University of North Carolina at Charlotte)

Sets

Note 1 (Set) *A set is a well-defined collection of objects.*

Notation:

- Constructive: $A = \{2, 3, 4, 5, 6\}$
- Descriptive: $B = \{x \quad | \quad x$ is an integer and $1 < x < 7\}$

Sets

Note 2 (Elements, Containement) *Things contained by sets are called* elements.

Notation (set S, element e):

- S contains e : $e \in S$
- S does not contain e : $e \notin S$

Examples:

- $1 \in \{0, 1, 2, 3\}$
- $7 \in \{x \quad | \quad x \text{ odd integer } \}$
- "hello" $\notin$ {"good bye", "good morning"}
- $45 \notin \{\}$

Sets

Observation 13 *A necessary requirement for* elements *is that they can be told apart. In other words that for two elements x and y, one can always decide whether $x = y$ is true or false.*

Observation 14 *The terms set, element and containment are not properly defined.*

Observation 15 *Mathematics can deal with sets without them being properly defined, just through axioms associated with their behavior.*

Sets

Observation 16 *The universe of every discourse is a set.*

Observation 17 *Sets cannot contain the same element multiple times.*

Observation 18 *There is no set-related meaning to the order of elements in a set.*

Notation:

$$\{2, 1, 4, 2\} \text{ is exactly the same thing as } \{1, 2, 4\}$$

Subsets

Definition 1 (Subset) *We say that a set A* is a Subset *(written $A \subseteq B$) of a set B iff:*

$$\forall x[x \in A \Rightarrow x \in B]$$

Definition 2 (Proper Subset) *We say that a set A* is a Proper Subset *(written $A \subset B$) of a set B iff:*

$$\forall x[x \in A \Rightarrow x \in B] \wedge \exists y[y \in b \wedge y \notin A]$$

Examples:

- $\{1, 4\} \subseteq \{0, 1, 3, 4\}$
- $\{1, 2\} \subseteq \{1, 2\}$
- $\neg[\{1, 2\} \subset \{1, 2\}]$

Set equality

Definition 3 *Two sets are equal iff they are subsets of each other.*

$$(A = B) \Leftrightarrow [(A \subseteq B) \wedge (B \subseteq A)]$$

Cardinality

Note 3 *The cardinality of a finite set is the number of elements in that set.*

Example:

- $|\{1, 2, 4, 8, 16\}| = 5$

Observation 19 *The notion of cardinality can also be extended to infinite sets*[a].

[a] Later discussion.

Empty set

Definition 4 *The* empty set *is a* *unique set* *with no elements (written $\emptyset$).*

Observation 20

$$|\emptyset| = 0$$

Theorem 1 *Consider a set $A \subseteq U$ (in a given universe). The following are true:*

1. $\emptyset \in A$
2. $A \neq \emptyset \Rightarrow \emptyset \subset A$

Subset transitivity

Theorem 2 *Consider A, B, C sets. The following are true:*

1. $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$
2. $A \subset B \wedge B \subset C \Rightarrow A \subset C$
3. $A \subset B \wedge B \subseteq C \Rightarrow A \subset C$
4. $A \subseteq B \wedge B \subset C \Rightarrow A \subset C$

Russel's paradox (exercise 27)

Observation 21 *The concept of set and considering whether a set can be a member of itself raise difficult question.*

Observation 22 (Russell's paradox) *Consider*
$S = \{A \quad | \quad A \text{ is a set } \wedge A \notin A\}$.

The following are true:

1. $S \in S \Rightarrow S \notin S$
2. $S \notin S \Rightarrow S \in S$