

**SFU MACM-101-D3 2004-2 week 4**

**Manuel Zahariev**  
**E-mail: manuelz@cs.sfu.ca**

**October 4, 2004**

*Revision : 1.8*

**Mathematical terminology**

definition statements (propositions)

- definition : a *labelling* or *naming* of a given entity or class of entities.
- axiom (postulate) : an utterance that is said to be a true statement.
- theorem : a statement that **can be proven true** by reasoning from axioms.
- corollary : a statement (theorem) that is a direct consequence of a notable theorem.
- conjecture : an utterance that is believed to be a true statement, but which hasn't been proven or disproven yet.

**Mathematical terminology**

For statements  $p$  and  $q$ :

- $p$  is sufficient for  $q$  ( $p \rightarrow q$ )
- $p$  is necessary for  $q$  ( $q \rightarrow p$ )
- $p$  is necessary and sufficient  $q$  ( $p \leftrightarrow q$ )
- $p$  if and only if  $q$  ( $p \leftrightarrow q$ )

**Mathematical terminology**

The form of definitions:

We **call** something  $L$  *if and only if*  $P$ .

**Observation 1** *Axioms, theorems are of the form:  $P$ , where  $P$  is a statement.*

**Observation 2** *Many propositions in mathematics are formulated as quantified open statements.*

**Observation 3** *Many theorems are formulated as logical implications or as logical equivalences (conjunctions of opposed logical implications).*

**Mathematical terminology**

**Observation 4** *The proof of theorems is the central activity in the study of mathematics.*

**Observation 5 (The life cycle of mathematical propositions)** • *create a new Conjecture.*

- *Conjecture proven false.*
- *Conjecture  $\rightsquigarrow$  theorem.*
- *Conjecture  $\rightsquigarrow$  axiom.*

**Observation 6** *The truth of axioms **cannot be demonstrated**, but is stated (or postulated).*

**Applications of Mathematics**

**Observation 7** *The application of theorems in specific cases, where reality fits the premise of the theorem.*

Example: finding your location using a map and a compass.

**Observation 8 (The rule of Universal Generalization)** *For universally quantified open statements, the truth of the open statement with a substitution of the variable with any given value in the universe will follow from the truth of the universally quantified open statement.*

## Mathematics

**Observation 9** The truth of a given universally quantified statement (conjecture) can be proven through *exhaustive coverage of the universe* (the method of exhaustion).

**Observation 10** The falsity of a universally quantified statement (conjecture) can be proven as a side effect of the method of exhaustion.

[http://www-gap.dcs.st-and.ac.uk/history/HistTopics/Fermat's\\_last\\_theorem.html](http://www-gap.dcs.st-and.ac.uk/history/HistTopics/Fermat's_last_theorem.html)

## Reasoning using quantified statements

**Observation 11** Use of the method of exhaustion can be very demanding or even impossible, depending on the size of the universe.

**Observation 12 (The rule of Universal Generalization)** Propositional logic can be applied to instantiations of universally quantified statements to *fixed, but arbitrary values*.

Example: Geometry

<http://www.math.uncc.edu/~droyster/courses/spring99/math3181/classnotes/axioms.pdf>  
(from Dr. David C. Royster, University of North Carolina at Charlotte)

## Sets

**Note 1 (Set)** A set is a well-defined *collection* of objects.

Notation:

- Constructive:  $A = \{2, 3, 4, 5, 6\}$
- Descriptive:  $B = \{x \mid x \text{ is an integer and } 1 < x < 7\}$

## Sets

**Note 2 (Elements, Containment)** Things *contained* by sets are called *elements*.

Notation (set  $S$ , element  $e$ ):

- $S$  contains  $e$ :  $e \in S$
- $S$  does not contain  $e$ :  $e \notin S$

Examples:

- $1 \in \{0, 1, 2, 3\}$
- $7 \in \{x \mid x \text{ odd integer}\}$
- "hello"  $\notin \{\text{"good bye"}, \text{"good morning"}\}$
- $45 \notin \{\}$

## Sets

**Observation 13** A necessary requirement for *elements* is that *they can be told apart*. In other words that for two elements  $x$  and  $y$ , one can always decide whether  $x = y$  is true or false.

**Observation 14** The terms *set*, *element* and *containment* are not properly defined.

**Observation 15** Mathematics can deal with sets without them being properly defined, just through axioms associated with their behavior.

## Sets

**Observation 16** The universe of every discourse is a set.

**Observation 17** Sets *cannot contain the same element multiple times*.

**Observation 18** There is no set-related meaning to the *order of elements* in a set.

Notation:

$\{2, 1, 4, 2\}$  is exactly the same thing as  $\{1, 2, 4\}$

### Subsets

**Definition 1 (Subset)** We say that a set  $A$  is a **Subset** (written  $A \subseteq B$ ) of a set  $B$  iff:

$$\forall x[x \in A \Rightarrow x \in B]$$

**Definition 2 (Proper Subset)** We say that a set  $A$  is a **Proper Subset** (written  $A \subset B$ ) of a set  $B$  iff:

$$\forall x[x \in A \Rightarrow x \in B] \wedge \exists y[y \in B \wedge y \notin A]$$

Examples:

- $\{1, 4\} \subseteq \{0, 1, 3, 4\}$
- $\{1, 2\} \subseteq \{1, 2\}$
- $\neg[\{1, 2\} \subset \{1, 2\}]$

### Set equality

**Definition 3** Two sets are equal iff they are subsets of each other.

$$(A = B) \Leftrightarrow [(A \subseteq B) \wedge (B \subseteq A)]$$

### Cardinality

**Note 3** The **cardinality** of a finite set is the number of elements in that set.

Example:

- $|\{1, 2, 4, 8, 16\}| = 5$

**Observation 19** The notion of cardinality can also be extended to **infinite sets**<sup>a</sup>.

<sup>a</sup>Later discussion.

### Empty set

**Definition 4** The **empty set** is a **unique set** with no elements (written  $\emptyset$ ).

**Observation 20**

$$|\emptyset| = 0$$

**Theorem 1** Consider a set  $A \subseteq U$  (in a given universe). The following are true:

1.  $\emptyset \in A$
2.  $A \neq \emptyset \Rightarrow \emptyset \subset A$

### Subset transitivity

**Theorem 2** Consider  $A, B, C$  sets. The following are true:

1.  $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$
2.  $A \subset B \wedge B \subset C \Rightarrow A \subset C$
3.  $A \subset B \wedge B \subseteq C \Rightarrow A \subset C$
4.  $A \subseteq B \wedge B \subset C \Rightarrow A \subset C$

### Russel's paradox (exercise 27)

**Observation 21** The concept of set and considering whether a set can be a member of itself raise difficult question.

**Observation 22 (Russell's paradox)** Consider

$$S = \{A \mid A \text{ is a set} \wedge A \notin A\}.$$

The following **are true**:

1.  $S \in S \Rightarrow S \notin S$
2.  $S \notin S \Rightarrow S \in S$