SFU MACM-101-D3 2004-2 week 6

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Counting

Note 1 (Counting finite sets) *The chapter on counting deals with the* counting of elements of finite sets *(cardinalities).*

Observation 1 (Cardinalities of infinite sets) *Cardinalities of infinite sets form an extension to this theory of counting. (outside of the scope of our coverage of counting).*

Rule of Sum

Definition 1 (Two disjoint sets) *Consider* A, B *sets. These sets are called* disjoint *iff* $A \cap B = \emptyset$.

Definition 2 (Disjoint sets: generalization) *Consider* $n \in \mathbb{N}$, $n \geq 2$ *and* n *sets* $A_1..A_n$. *The sets* $A_1..A_n$ *are called* disjoint *iff*

$$\forall i, j \in 1..n \ i \neq j \rightarrow A_i \cap A_j = \emptyset$$

Rule of Sum

Observation 2 (Base case for the rule of sum) *Consider A, B disjoint sets. Then $|A \cup B| = |A| + |B|$.*

Proof: direct, using week 5: Observation 11.

Theorem 1 (Rule of sum) *For any n disjoint sets A_k, where $k \in 1..n$*

$$|\bigcup_{k=1}^{n} A_k| = \sum_{k=1}^{n} |A_k|$$

Rule of Sum

Observation 3 (Rule of sum) *If a task can be performed in m ways and another task can be performed in n ways, performing either task in either way can be done $m + n$ ways.*

Read:

- task $\rightsquigarrow$ set
- presuppose: tasks are distinct $\rightsquigarrow$ disjoint sets
- ways $\rightsquigarrow$ cardinality
- either task either way $\rightsquigarrow$ union

Rule of Sum

Observation 4 *The sets $A - B$, $A \cap B$, $B - A$ are disjoint.*

Observation 5

$$|A \cup B| = |A - B| + |A \cap B| + |B - A|$$

Cartesian product

Definition 3 (Pair) *Given two sets A and B, a* pair *is an ordered sequence of elements (a, b), where $a \in A \ \wedge \ b \in B$.*

Definition 4 (Cartesian product of two sets) *Given two sets, A and B, the* Cartesian product of A and B *(des Cartes: French mathematician) is a set:*

$$A \times B = \{(a, b) \ pair \mid a \in A \ \wedge \ b \in B\}$$

Cartesian product

Definition 5 (Tuple) *Given* $n \geq 2$ *sets* $A_1..A_n$, *a* n-tuple *is an ordered sequence of elements* $(a_1, a_2, ..., a_n)$ *where* $\forall i \in 1..n\ a_i \in A_i$.

Definition 6 (Cartesian product) *Given* $n \geq 2$ *sets* $A_1..A_n$, *the Cartesian product of these sets is a set:*

$$\prod_{i=1}^{n} A_i = \{(a_1, a_2, ..., a_n) \mid \forall i \in 1..n\ a_i \in A_i\}$$

Rule of Product

Theorem 2 (Rule of product for two sets)

$$|A \times B| = |A| \times |B|$$

Proof: involves mathematical induction.

Theorem 3 (Rule of product)

$$|\prod_{i=1}^{n} A_i| = \prod_{i=1}^{n} |A_i|$$

Observation 6 *The rule of product assumes the ability to choose independently elements from each of the source sets into the destination tuple.*

Factorial

Definition 7 (Factorial (recursive definition)) *Given* $n \in \mathbb{N}$*,* $n!$ *(n factorial) is:*

$$n! = \begin{cases} 1 & \textit{if } n = 0 \quad (0! = 1) \\ n \times (n-1)! & \textit{if } n > 0 \end{cases}$$

Observation 7

$$0! = 1$$

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$$

Linear arrangements

Definition 8 (Linear arrangement) *Given a finite set A, such that $|A| = n$, and given $k \in 1..n$, a* linear arrangement *is:*

- *a k-tuple $(a_1, a_2, \ldots, a_k) \in A^k$*
- *where $\forall i \in 1..k\ a_i \in A$*
- *and where all a_i are distinct ($\forall i\ \forall j\ [i \neq j \rightarrow a_i \neq a_j]$)*

Permutations

Question: how many linear arrangements of size n of a set with cardinality n are there?

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Answer: *n-permutations of n* ($P(n,n) = n!$)

Permutations

Definition 9 (Permutations) *Given a finite set A, such that $|A| = n$, and given $k \in 1..n$, the* k-permutations of the set A *represent the number of all linear arrangements of size k of the set A (written $P(n, k)$).*

Observation 8

$$P(n,k) = |\{(a_1, a_2, \ldots, a_k) \in A^k \mid \forall i\, \forall j\ i \neq j \rightarrow a_i \neq a_j\}| = \frac{n!}{(n-k)!}$$

Proof: use the rule of product.

Example: the number of possible results of shuffling a deck of n cards.

Linear arrangements with indistinguishable symbols

Definition 10 (Arrangements equality) *Given a finite set A, such that $|A| = n$, and given $k \in 1..n$, consider r disjoint subsets of A: $A_1, A_2, \ldots A_r$ ($\forall i \in 1..r\ A_i \subset A$).*

Any two arrangements (with indistinguishable symbols) with the following properties are considered equal*:*

- *first arrangement:* $(a_1, a_2, \ldots, a_k)$
- *second arrangement:* $(b_1, b_2, \ldots, b_k)$
- $\forall j \in 1..k\ [(\forall i\ i \in 1..r\ a_j \notin A_i) \rightarrow (a_j = b_j)]$
- $\forall j \in 1..k\ \exists i \in 1..r\ [(a_j \in A_i) \rightarrow (b_j \in A_i)]$

Linear arrangements with indistinguishable symbols

Observation 9 *The elements of each of the r disjoint subsets of A $(A_1, ...A_r)$ are considered* indistinguishable symbols*.*

Example: the result of shuffling a deck of n cards, some of them undistinguishable "Jokers".

Permutations with indistinguishable symbols

Definition 11 (Permutations with indistinguishable symbols) *Given a finite set A, such that $|A| = n$, and given $k \in 1..n$, consider r disjoint subsets of A: $A_1, A_2, \ldots A_r$ ($\forall i \in 1..r \; A_i \subset A$). The* k-permutations with indistinguishable symbols *is the cardinality of the set of arrangements with indistinguishable symbols of size k:* $I(n, k, A_1, A_2, \ldots, A_r)$

Permutations with indistinguishable symbols

Observation 10 (n-Permutations with indistinguishable symbols) *Cosider* $\forall i \in 1..r\ |A_i| = n_i$.

$$I(n, n, A_1, A_2, \ldots, A_r) =$$

$$= |\{z \mid z \text{ is an arrangement with indistinguishable symbols of size } n\}| =$$

$$= \frac{n!}{\prod_{i=1}^{r} |A_i|!} = \frac{n!}{\prod_{i=1}^{r} n_i!} =$$

Observation 11 *The formula for k-Permutations with indistinguishable symbols is more complicated and is outside of the scope of this course.*

Combinations

Definition 12 (k-Combinations) *Consider A a finite set, $n = |A|$ and $k \in 0..n$.* k-combinations of n *(also called* n-choose-k*) represent the cardinality of the set of all subsets of cardinality k of the set A.*

$$C(n,k) = \binom{n}{k} = |\{S \mid S \subset A \ \wedge \ |S| = k\}|$$

Example: number of possible 6/49 plays: $C(49, 6)$.

Combinations

Observation 12 *The only difference between k-permutations and k-combinations is the difference between* linear arrangements *and* sets. *For sets, the order of elements does not count.*

Observation 13

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)! \cdot k!}$$

Counting: summary

What you need to remember (formulae and examples) so far:

- When calculating totals, you can only add the number of elements of disjoint sets (*rule of sum*). Example: add the number of votes for Bush, Kerry and Nader to get the total number of valid votes.

- You can multiply the number of elements in sets (*rule of product*), when the choices of elements of each set are unrelated. Example: the number of license plates of the type "ABC 123".

Counting: summary

- *n-permutations*: $P(n,n) = n!$. Example: number of possible outcomes of shuffling a deck of 52 different cards.
- *n-permutations with classes of indistinguishable elements*: $I(n,n,A_1,\ldots,A_r) = \frac{n!}{n_1!\cdot n_2!\ldots n_r!}$. Example: number of possible "words" using the letters: R, R, O, L, L, E.
- *k-permutations*: $P(n,k) = \frac{n!}{(n-k)!}$. Example: number of possible outcomes of shuffling a deck of 52 different cards, then taking the first 5 of them (when order counts).
- *k-combinations*: $C(n,k) = \frac{n!}{(n-k)!\cdot k!}$. Example: number of possible plays for 6/49.