

SFU MACM-101-D3 2004-2 week 6

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Counting

Note 1 (Counting finite sets) *The chapter on counting deals with the counting of elements of finite sets (cardinalities).*

Observation 1 (Cardinalities of infinite sets) *Cardinalities of infinite sets form an extension to this theory of counting. (outside of the scope of our coverage of counting).*

Rule of Sum

Definition 1 (Two disjoint sets) *Consider A, B sets. These sets are called disjoint iff $A \cap B = \emptyset$.*

Definition 2 (Disjoint sets: generalization) *Consider $n \in \mathbb{N}$, $n \geq 2$ and n sets $A_1..A_n$. The sets $A_1..A_n$ are called disjoint iff*

$$\forall i, j \in 1..n \ i \neq j \rightarrow A_i \cap A_j = \emptyset$$

Rule of Sum

Observation 2 (Base case for the rule of sum) *Consider A, B disjoint sets. Then $|A \cup B| = |A| + |B|$.*

Proof: direct, using week 5: Observation 11.

Theorem 1 (Rule of sum) *For any n disjoint sets A_k , where $k \in 1..n$*

$$|\bigcup_{k=1}^n A_k| = \sum_{k=1}^n |A_k|$$

Rule of Sum

Observation 3 (Rule of sum) *If a task can be performed in m ways and another task can be performed in n ways, performing either task in either way can be done $m + n$ ways.*

Read:

- task \rightsquigarrow set
- presuppose: tasks are distinct \rightsquigarrow disjoint sets
- ways \rightsquigarrow cardinality
- either task either way \rightsquigarrow union

Rule of Sum

Observation 4 *The sets $A - B$, $A \cap B$, $B - A$ are disjoint.*

Observation 5

$$|A \cup B| = |A - B| + |A \cap B| + |B - A|$$

Cartesian product

Definition 3 (Pair) Given two sets A and B , a **pair** is an *ordered sequence* of elements (a, b) , where $a \in A \wedge b \in B$.

Definition 4 (Cartesian product of two sets) Given two sets, A and B , the **Cartesian product of A and B** (des Cartes: French mathematician) is a set:

$$A \times B = \{(a, b) \text{ pair} \mid a \in A \wedge b \in B\}$$

Cartesian product

Definition 5 (Tuple) Given $n \geq 2$ sets $A_1..A_n$, a **n -tuple** is an ordered sequence of elements (a_1, a_2, \dots, a_n) where $\forall i \in 1..n \ a_i \in A_i$.

Definition 6 (Cartesian product) Given $n \geq 2$ sets $A_1..A_n$, the Cartesian product of these sets is a set:

$$\prod_{i=1}^n A_i = \{(a_1, a_2, \dots, a_n) \mid \forall i \in 1..n \ a_i \in A_i\}$$

Rule of Product

Theorem 2 (Rule of product for two sets)

$$|A \times B| = |A| \times |B|$$

Proof: involves mathematical induction.

Theorem 3 (Rule of product)

$$\left| \prod_{i=1}^n A_i \right| = \prod_{i=1}^n |A_i|$$

Observation 6 The rule of product assumes the *ability to choose independently* elements from each of the source sets into the destination tuple.

Factorial

Definition 7 (Factorial (recursive definition)) Given $n \in \mathbb{N}$, $n!$ (*n factorial*) is:

$$n! = \begin{cases} 1 & \text{if } n = 0 \quad (0! = 1) \\ n \times (n-1)! & \text{if } n > 0 \end{cases}$$

Observation 7

$$0! = 1$$

$$n! = 1 \times 2 \times 3 \times \dots \times (n-1) \times n$$

Linear arrangements

Definition 8 (Linear arrangement) Given a finite set A , such that $|A| = n$, and given $k \in 1..n$, a **linear arrangement** is:

- a k -tuple $(a_1, a_2, \dots, a_k) \in A^k$
- where $\forall i \in 1..k \ a_i \in A$
- and where all a_i are distinct ($\forall i \neq j \ [i \neq j \rightarrow a_i \neq a_j]$)

Permutations

Question: how many linear arrangements of size n of a set with cardinality n are there?

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Answer: n -permutations of n ($P(n, n) = n!$)

Permutations

Definition 9 (Permutations) Given a finite set A , such that $|A| = n$, and given $k \in 1..n$, the k -permutations of the set A represent the number of all linear arrangements of size k of the set A (written $P(n, k)$).

Observation 8

$$P(n, k) = |\{(a_1, a_2, \dots, a_k) \in A^k \mid \forall i \forall j \ i \neq j \rightarrow a_i \neq a_j\}| = \frac{n!}{(n-k)!}$$

Proof: use the rule of product.

Example: the number of possible results of shuffling a deck of n cards.

Linear arrangements with indistinguishable symbols

Definition 10 (Arrangements equality) Given a finite set A , such that $|A| = n$, and given $k \in 1..n$, consider r disjoint subsets of A : A_1, A_2, \dots, A_r ($\forall i \in 1..r \ A_i \subset A$).

Any two arrangements (with indistinguishable symbols) with the following properties are considered equal:

- first arrangement: (a_1, a_2, \dots, a_k)
- second arrangement: (b_1, b_2, \dots, b_k)
- $\forall j \in 1..k \ [(\forall i \ i \in 1..r \ a_j \notin A_i) \rightarrow (a_j = b_j)]$
- $\forall j \in 1..k \ \exists i \in 1..r \ [(a_j \in A_i) \rightarrow (b_j \in A_i)]$

Linear arrangements with indistinguishable symbols

Observation 9 The elements of each of the r disjoint subsets of A (A_1, \dots, A_r) are considered indistinguishable symbols.

Example: the result of shuffling a deck of n cards, some of them indistinguishable "Jokers".

Permutations with indistinguishable symbols

Definition 11 (Permutations with indistinguishable symbols)

Given a finite set A , such that $|A| = n$, and given $k \in 1..n$, consider r disjoint subsets of A : A_1, A_2, \dots, A_r ($\forall i \in 1..r \ A_i \subset A$). The k -permutations with indistinguishable symbols is the cardinality of the set of arrangements with indistinguishable symbols of size k : $I(n, k, A_1, A_2, \dots, A_r)$

Permutations with indistinguishable symbols

Observation 10 (n -Permutations with indistinguishable symbols)

Consider $\forall i \in 1..r \ |A_i| = n_i$.

$$\begin{aligned} I(n, n, A_1, A_2, \dots, A_r) &= \\ &= |\{z \mid z \text{ is an arrangement with indistinguishable symbols of size } n\}| = \\ &= \frac{n!}{r} = \frac{n!}{\prod_{i=1}^r |A_i|!} = \frac{n!}{\prod_{i=1}^r n_i!} \end{aligned}$$

Observation 11 The formula for k -Permutations with indistinguishable symbols is more complicated and is outside of the scope of this course.

Combinations

Definition 12 (*k*-Combinations) Consider A a finite set, $n = |A|$ and $k \in 0..n$. *k-combinations of n* (also called *n-choose-k*) represent the *cardinality of the set of all subsets of cardinality k of the set A* .

$$C(n, k) = \binom{n}{k} = |\{S \mid S \subset A \wedge |S| = k\}|$$

Example: number of possible 6/49 plays: $C(49, 6)$.

Combinations

Observation 12 The only difference between *k-permutations* and *k-combinations* is the difference between *linear arrangements* and *sets*. For sets, *the order of elements does not count*.

Observation 13

$$C(n, k) = \frac{P(n, k)}{k!} = \frac{n!}{(n-k)! \cdot k!}$$

Counting: summary

What you need to remember (formulae and examples) so far:

- When calculating totals, you can only add the number of elements of *disjoint sets* (*rule of sum*). Example: add the number of votes for Bush, Kerry and Nader to get the total number of valid votes.
- You can multiply the number of elements in sets (*rule of product*), when the choices of elements of each set are *unrelated*. Example: the number of license plates of the type "ABC 123".

Counting: summary

- *n-permutations*: $P(n, n) = n!$. Example: number of possible outcomes of shuffling a deck of 52 different cards.
- *n-permutations with classes of indistinguishable elements*: $I(n, n, A_1, \dots, A_r) = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_r!}$. Example: number of possible "words" using the letters: R, R, O, L, L, E.
- *k-permutations*: $P(n, k) = \frac{n!}{(n-k)!}$. Example: number of possible outcomes of shuffling a deck of 52 different cards, then taking the first 5 of them (when order counts).
- *k-combinations*: $C(n, k) = \frac{n!}{(n-k)! \cdot k!}$. Example: number of possible plays for 6/49.