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Well ordered sets

Definition 1 (Well ordering) *Consider a set U and an order relation $\leq$ between elements of that universe. U is said to be* well-ordered *iff:*

$$\forall S \subset U \; S \neq \emptyset \rightarrow \exists x \in S \; \forall y \in S x \leq y$$

*(every subset of U has a smallest [*minimum*] element).*

Observation 1 *$\emptyset$ is well ordered.*

Proof: the hypothesis is always false.

Properties of well ordering

Observation 2 *Consider a well-ordered set A. Every subset of A: $B \subset A$ is well ordered.*

Proof (contradiction): Consider $B \subset A$ so that B is not well ordered.

$\exists S \neq \emptyset$ proper subset of B that doesn't have a smallest element.

$$S \subset B \wedge \underbrace{B \subset A}_{\text{hypothesis}} \Rightarrow S \subset A$$

It follows that $\exists S \neq \emptyset$ subset of A that doesn't have a smallest element $\Rightarrow$ A is not a well ordered set (contradiction).

Well ordered sets of numbers

Observation 3 (Well ordering principle) *$\mathbb{Q}$ is not well ordered.*

Proof: contradiction.

Consider $\frac{p}{q} \in \mathbb{Q}^+$. The set $\{x \in \mathbb{Q} \mid x > \frac{p}{q}\}$ is not well ordered.

Observation 4 (Well ordering principle) *$\mathbb{R}$ is not well ordered.*

Proof: contradiction. $\mathbb{Q} \subset \mathbb{R}$ is not well ordered.

Well ordered sets of numbers

Axiom 1 (Well ordering principle) *$\mathbb{Z}^+$ is well ordered.*

Observation 5 *$\mathbb{N}$ is well ordered.*

Proof: direct; $\mathbb{N} = \mathbb{Z}^+$

Mathematical Induction

Theorem 1 *Mathematical Induction Consider $p(k)$ a predicate defined over the universe $\mathbb{N}$.*

$$\forall k \; p(k) \Leftrightarrow \begin{cases} p(0) \\ \forall k \; p(k) \rightarrow p(k+1) \end{cases}$$

Proof: contradiction, using the well ordering property of $\mathbb{N}$.