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Manuel Zahariev

E-mail: manuelz@cs.sfu.ca

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Cartesian product

Definition 1 (Tuple) *Given $n \geq 2$ sets $A_1..A_n$, a* n-tuple *is an ordered sequence of elements $(a_1, a_2, ..., a_n)$ where $\forall i \in 1..n\ a_i \in A_i$.*

Definition 2 (Cartesian product) *Given $n \geq 2$ sets $A_1..A_n$, the Cartesian product of these sets is a set:*

$$\prod_{i=1}^{n} A_i = \{(a_1, a_2, ..., a_n) \mid \forall i \in 1..n\ a_i \in A_i\}$$

Binary relations

Definition 3 *Considering A, B sets, any subset $\mathcal{R} \subseteq A \times B$ is called a* binary relation from A to B.

Examples:

- Every set of points in the plane is a relation from $\mathbb{R}$ to $\mathbb{R}$.
- $\leq$ is a relation from $\mathbb{Z}$ to $\mathbb{Z}$:
  $\leq \ = \{(x, x+a) \mid \forall x \ \forall a \ x \in \mathbb{Z} \ \wedge \ a \in \mathbb{N}\}$
- $\leq$ is a relation from $\mathbb{R}$ to $\mathbb{R}$:
  $\leq \ = \{(x, x+a) \mid \forall x \ \forall a \ x \in \mathbb{R} \ \wedge \ a \in \mathbb{R}^+\}$
- Equality.
- Friendship.

Relations

Notation: $(x, y) \in \mathcal{R} \Leftrightarrow x\mathcal{R}y$

Observation 1 *The number of all possible relations from a finite set A to a finite set B is $2^{|A \times B|} = 2^{|A||B|}$.*

Proof:

- use the rule of product (cardinality of $|A \times B|$)
- calculate the number of subsets of $A \times B$.

Functions

Definition 4 *Considering A and B sets, a* function f from A to B *(written $f : A \to B$) is a relation from A (*domain*) to B (*codomain*), where each element in A appears in exactly one pair.*

($\forall a \in A \; |\{(a, x) \mid x \in B\}| = 1$ *)*

Notation: $(x, y) \in f \Leftrightarrow xfy \Leftrightarrow y = f(x)$ (last most common)

Images under Functions

Definition 5 *Considering $f : A \to B$ a function and $C \subseteq A$, the* image of C under f *(written $f(C)$) is:*

$$f(C) = \{y \in B \mid \exists x \in A \ f(x) = y\}$$

Observation 2 *If $f : A \to B$ a function and two subsets of A, $C, D \subseteq A$ then:*

$$f(C \cup D) = f(C) \cup f(D)$$

Proof: using the definitions of union and image.

Observation 3 *If $f : A \to B$ a function and a finite subset of A, $C \subseteq A$, then $|f(C)| \leq |C|$*

Proof: induction by the cardinality of C.

Functions

Observation 4 *The number of all possible functions from a finite set A to a finite set B is $|B|^{|A|}$.*

Proof:

- Same as asking "how many $|A|$-tuples of elements from B there are?".
- Use the rule of product to calculate $|\prod_{k=1}^{|A|} B| = |B|^{|A|}$

Observation 5

$$\forall m, n \in \mathbb{N}^* \; n^m < 2^{mn}$$

Injective functions

Definition 6 *A function $f : A \longrightarrow B$ is said to be* injective *(a.k.a.* one-to-one *or* injection*) iff:*

$$\forall x, y \in A \; f(x) = f(y) \Rightarrow x = y$$

(Every $y \in B$ has at most one *$x \in A$ that translate to y through f).*

Observation 6 *If $f : A \longrightarrow B$ is an injection and A is a finite set, $|f(A)| = |A|$.*

Proof: induction on subsets $A_n \subseteq A$ with $|A_n| = n$

Observation 7

$$\exists f : A \longrightarrow B \; \textit{injection} \; \Leftrightarrow |A| \leq |B|$$

Proof (for finite sets): direct, from previous observation.

Injective functions

Observation 8 *There are exactly* $P(|B|, |A|) = \frac{|B|!}{(|B|-|A|)!}$ *injections from* A *to* B *(both finite sets) where* $|A| \leq |B|$.

Proof: same the number of sequences ($|A|$-tuples) of $|B|$ unique elements from A.

Image of intersection

Observation 9 *If $f : A \to B$ is a function and two finite subsets of A, $C, D \subseteq A$, then:*

$$f(C \cap D) \subseteq f(C) \cap f(D)$$

$$|f(C \cap D)| \leq |f(C) \cap f(D)|$$

Proof: using the definition of function image and intersection.

Observation 10 *If $f : A \to B$ is an injective function and two finite subsets of A, $C, D \subseteq A$, then:*

$$|f(C \cap D)| = |f(C) \cap f(D)|$$

Proof: proof needed only for $\geq$.

Surjective functions

Definition 7 *A function $f : A \to B$ is said to be* surjective *(a.k.a.* onto *or* surjection*) iff:*

$$\forall y \in B \; \exists x \in A \; f(x) = y$$

(Every $y \in B$ has at least one $x \in A$ that translate to y through f).

Observation 11 *If $f : A \to B$ is a surjection, then $f(A) = B$.*

Observation 12

$$\exists f : A \to B \textit{ surjection } \Leftrightarrow |A| \geq |B|$$

Bijective functions

Definition 8 *A function $f : A \rightarrow B$ is said to be* bijective *(a.k.a.* bijection*) iff:*

1. *f is an injection, and*
2. *f is a surjection*

Observation 13

$$\exists f : A \rightarrow B \text{ } \textit{bijection} \text{ } \Leftrightarrow |A| = |B|$$

Proof: immediate; $\geq$ injection and $\leq$ surjection.