

SFU MACM-101-D3 2004-2 week 13

Manuel Zahariev
E-mail: manuelz@cs.sfu.ca

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Cartesian product

Definition 1 (Tuple) Given $n \geq 2$ sets $A_1..A_n$, a **n -tuple** is an ordered sequence of elements (a_1, a_2, \dots, a_n) where $\forall i \in 1..n$ $a_i \in A_i$.

Definition 2 (Cartesian product) Given $n \geq 2$ sets $A_1..A_n$, the Cartesian product of these sets is a set:

$$\prod_{i=1}^n A_i = \{(a_1, a_2, \dots, a_n) \mid \forall i \in 1..n$$

Binary relations

Definition 3 Considering A, B sets, any subset $\mathcal{R} \subseteq A \times B$ is called a **binary relation from A to B** .

Examples:

- Every set of points in the plane is a relation from \mathbb{R} to \mathbb{R} .
- \leq is a relation from \mathbb{Z} to \mathbb{Z} :
 $\leq = \{(x, x+a) \mid \forall x \forall a$ $x \in \mathbb{Z} \wedge a \in \mathbb{N}\}$
- \leq is a relation from \mathbb{R} to \mathbb{R} :
 $\leq = \{(x, x+a) \mid \forall x \forall a$ $x \in \mathbb{R} \wedge a \in \mathbb{R}^+\}$
- Equality.
- Friendship.

Relations

Notation: $(x, y) \in \mathcal{R} \Leftrightarrow x\mathcal{R}y$

Observation 1 The number of all possible relations from a finite set A to a finite set B is $2^{|A \times B|} = 2^{|A||B|}$.

Proof:

- use the rule of product (cardinality of $|A \times B|$)
- calculate the number of subsets of $A \times B$.

Functions

Definition 4 Considering A and B sets, a **function f from A to B** (written $f : A \rightarrow B$) is a relation from A (domain) to B (codomain), where each element in A appears in exactly one pair.

 $(\forall a \in A \mid \{(a, x) \mid x \in B\} = 1)$
Notation: $(x, y) \in f \Leftrightarrow xfy \Leftrightarrow y = f(x)$ (last most common)

Images under Functions

Definition 5 Considering $f : A \rightarrow B$ a function and $C \subseteq A$, the **image of C under f** (written $f(C)$) is:

$$f(C) = \{y \in B \mid \exists x \in A$$

Observation 2 If $f : A \rightarrow B$ a function and two subsets of A , $C, D \subseteq A$ then:

$$f(C \cup D) = f(C) \cup f(D)$$

Proof: using the definitions of union and image.

Observation 3 If $f : A \rightarrow B$ a function and a finite subset of A , $C \subseteq A$, then $|f(C)| \leq |C|$

Proof: induction by the cardinality of C .

Functions

Observation 4 The number of all possible functions from a finite set A to a finite set B is $|B|^{|A|}$.

Proof:

- Same as asking “how many $|A|$ -tuples of elements from B there are?”.

- Use the rule of product to calculate $\prod_{k=1}^{|A|} |B| = |B|^{|A|}$

Observation 5

$$\forall m, n \in \mathbb{N}^* \quad n^m < 2^{mn}$$

Injective functions

Definition 6 A function $f : A \rightarrow B$ is said to be **injective** (a.k.a. **one-to-one** or **injection**) iff:

$$\forall x, y \in A \quad f(x) = f(y) \Rightarrow x = y$$

(Every $y \in B$ has **at most one** $x \in A$ that translate to y through f).

Observation 6 If $f : A \rightarrow B$ is an injection and A is a finite set, $|f(A)| = |A|$.

Proof: induction on subsets $A_n \subseteq A$ with $|A_n| = n$

Observation 7

$$\exists f : A \rightarrow B \text{ injection} \Leftrightarrow |A| \leq |B|$$

Proof (for finite sets): direct, from previous observation.

Injective functions

Observation 8 There are exactly $P(|B|, |A|) = \frac{|B|!}{(|B|-|A|)!}$ injections from A to B (both finite sets) where $|A| \leq |B|$.

Proof: same the number of sequences ($|A|$ -tuples) of $|B|$ unique elements from A .

Image of intersection

Observation 9 If $f : A \rightarrow B$ is a function and two finite subsets of A , $C, D \subseteq A$, then:

$$f(C \cap D) \subseteq f(C) \cap f(D)$$

$$|f(C \cap D)| \leq |f(C) \cap f(D)|$$

Proof: using the definition of function image and intersection.

Observation 10 If $f : A \rightarrow B$ is an **injective** function and two finite subsets of A , $C, D \subseteq A$, then:

$$|f(C \cap D)| = |f(C) \cap f(D)|$$

Proof: proof needed only for \geq .

Surjective functions

Definition 7 A function $f : A \rightarrow B$ is said to be **surjective** (a.k.a. **onto** or **surjection**) iff:

$$\forall y \in B \quad \exists x \in A \quad f(x) = y$$

(Every $y \in B$ has **at least one** $x \in A$ that translate to y through f).

Observation 11 If $f : A \rightarrow B$ is a surjection, then $f(A) = B$.

Observation 12

$$\exists f : A \rightarrow B \text{ surjection} \Leftrightarrow |A| \geq |B|$$

Bijjective functions

Definition 8 A function $f : A \rightarrow B$ is said to be **bijjective** (a.k.a. **bijection**) iff:

1. f is an injection, and
2. f is a surjection

Observation 13

$$\exists f : A \rightarrow B \text{ bijection} \Leftrightarrow |A| = |B|$$

Proof: immediate; \geq injection and \leq surjection.