

Duality for Min-Max Orderings and Dichotomy for Min Cost Homomorphisms

Pavol Hell and Arash Rafiey
School of Computing Science
Simon Fraser University
Burnaby, B.C., Canada, V5A 1S6 *

Abstract

Min-Max orderings correspond to conservative lattice polymorphisms. Digraphs with Min-Max orderings have polynomial time solvable minimum cost homomorphism problems. They can also be viewed as digraph analogues of proper interval graphs and bigraphs.

We give a forbidden structure characterization of digraphs with a Min-Max ordering which implies a polynomial time recognition algorithm. We also similarly characterize digraphs with an extended Min-Max ordering, and we apply this characterization to prove a conjectured form of dichotomy for minimum cost homomorphism problems.

1 Introduction

Let H be any digraph. A linear ordering $<$ of $V(H)$ is a *Min-Max ordering* if $i < j, s < r$ and $ir, js \in A(H)$ imply that $is \in A(H)$ and $jr \in A(H)$.

Min-Max orderings correspond to a particular type of lattice polymorphisms [8]. For digraphs G and H , a mapping $f : V(G) \rightarrow V(H)$ is a *homomorphism of G to H* if $f(u)f(v)$ is an arc of H whenever uv is an arc of G [24]. The *product* $G \times H$ of digraphs G and H has the vertex set $V(G) \times V(H)$ and there is an arc in $G \times H$ from (u, v) to (u', v') if G has an arc from u to u' and H has an arc from v to v' . The *power* H^k is recursively defined as $H^1 = H$ and $H^{k+1} = H \times H^k$. A *polymorphism* of H is a homomorphism $f : H^k \rightarrow H$, for some positive integer k . Polymorphisms are of interest in the solution of constraint satisfaction problems [9, 26]. We say that polymorphisms $f, g : H^2 \rightarrow H$ are *lattice polymorphisms* of H , if each f and g is an associative, commutative, and idempotent,

*pavol@cs.sfu.ca , arashr@cs.sfu.ca

and if moreover f and g satisfy the absorption identities $f(u, g(u, v)) = g(u, f(u, v)) = u$. It is easy to see that the usual operations of minimum $f(u, v) = \min(u, v)$ and maximum $g(u, v) = \max(u, v)$, with respect to a fixed linear ordering $<$, are polymorphisms if and only if $<$ is a Min-Max ordering. It is also clear that they satisfy the lattice axioms. Thus a digraph which has a Min-Max ordering does admit lattice polymorphisms. In fact, a digraph admits a Min-Max ordering if and only if it admits lattice polymorphisms f, g that are *conservative*, i.e., satisfy $f(u, v) \in \{u, v\}, g(u, v) \in \{u, v\}$. Thus we are describing a forbidden structure characterization (and a polynomial time recognition algorithm) of digraphs with conservative lattice polymorphisms.

An undirected graph (viewed as a symmetric digraph) admits a Min-Max ordering if and only if each component is either a reflexive proper interval graph or an irreflexive proper interval bigraph [16]. Thus digraphs with Min-Max orderings can be viewed as digraph analogues of proper interval graphs. Proper interval graphs (and bigraphs) are characterized by simple forbidden structures, and recognized in polynomial time [29]. In this paper, we give a polynomial characterization of digraphs with a Min-Max ordering, suggesting that these digraph analogues also have interesting structure. Our characterization is in terms of a novel forbidden structure, which we call a symmetrically invertible pair. We call our characterization ‘duality’ in the broad sense of having the presence of some structure (Min-Max ordering) certified by the absence of some other (forbidden) structure.

We give a similar characterization of digraphs with certain extended Min-Max orderings, of interest in minimum cost homomorphism problems. The *minimum cost homomorphism problem* for H , denoted $\text{MinHOM}(H)$, asks whether or not an input digraph G , with integer costs $c_i(u), u \in V(G), i \in V(H)$, and an integer k , admits a homomorphism to H of total cost $\sum_{u \in V(G)} c_{f(u)}(u)$ not exceeding k . The problem $\text{MinHOM}(H)$ was first formulated in [20]; it unifies and generalizes several other problems [22, 25, 27, 28, 30], including two other well studied homomorphism problems, the problem $\text{HOM}(H)$ asking for just the existence of homomorphisms [23], and the problem $\text{ListHOM}(H)$ asking for the existence of homomorphisms in which vertices of G map to vertices of H on allowed lists [12]. For undirected graphs H , the complexity of both problems has been classified [23, 12], and so has the complexity of the problem $\text{MinHOM}(H)$ [16]. In each case, the classification is a dichotomy, in the sense that each problem $\text{HOM}(H)$ is polynomial time solvable or NP-complete. For digraphs, the dichotomy of $\text{HOM}(H)$ is an important unproved conjecture, equivalent to the so-called CSP Dichotomy Conjecture [14, 7]. Recent progress specifically on classifying the complexity of $\text{HOM}(H)$ for classes of digraphs H was reported in [4, 5]. A simple dichotomy classification of $\text{ListHOM}(H)$ for reflexive digraphs is described in [13]; for general digraphs dichotomy follows from more the general results in [6]. A simple dichotomy classification of $\text{MinHOM}(H)$ for reflexive digraphs can be found in [15]. It follows from [16, 15] that both for symmetric digraphs (undirected graphs) and for reflexive digraphs, $\text{MinHOM}(H)$ is polynomial time solvable if H admits a Min-Max ordering, and is NP-complete otherwise. This is not the case for general di-

graphs, as certain extended Min-Max orderings (defined in a later section) also imply a polynomial time algorithm [18]. However, it was conjectured in [18] that $\text{MinHOM}(H)$ is NP-complete unless H admits an extended Min-Max ordering. Several special cases of the conjecture have been verified [15, 16, 17, 18, 19]. We apply our characterization of digraphs with extended Min-Max ordering to prove this conjecture, obtaining a simple dichotomy classification of the minimum cost homomorphism problems in digraphs.

As can be expected, one can define minimum cost homomorphism problems for homomorphisms of more general relational structures H (instead of just one binary relation, H may have a finite number of finitary relations). In [11], the authors define, for each relational structure H , such a minimum cost constraint satisfaction problem $\text{MinCSP}(H)$. Even more generally, in [10], the authors define ‘soft’ constraint satisfaction problems, where each hard constraint (of preserving a k -ary relation) is replaced by a cost function assigning a cost to mapping any k -tuple to any other k -tuple. Thus $\text{MinCSP}(H)$ problems can be thought of as having soft unary constraints, with the other constraints being ‘hard’. Our results can be directly extended to relational structures H containing any number of binary relations. On the other hand, it follows from work of A. Bulatov (personal communication) that if dichotomy of MinHOM holds for structures with binary relations, then it holds for all structures. Another proof of dichotomy of MinCSP problems (but not of our simple classification) has recently been announced in [1].

2 Min-Max Orderings

If $uv \in E(H)$, we say that uv is an arc of H , or that uv is a *forward arc* of H ; we also say that vu is a *backward arc* of H . In any event, we say that u, v are *adjacent* in H if uv is a forward or a backward arc of H (and we often use *arc* in this more general sense). The *net length* of a walk is the number of forward arcs minus the number of backward arcs. (Note that a walk has a designated first and last vertex. For a closed walk we may always choose a direction in which the net length is non-negative.) An *oriented walk* is a walk in which each consecutive arc is *either* a forward arc *or* a backward arc. A digraph is *balanced* if it does not contain an oriented cycle of non-zero net length. It is easy to see that a digraph is balanced if and only if it admits a *labeling* of vertices by non-negative integers so that each arc goes from some level i to the level $i + 1$. The *height* of H is the maximum net length of a walk in H . Note that an unbalanced digraph has infinite height, and the height of a balanced digraph is the greatest label in a non-negative labeling in which some vertex has label zero.

For any walk $P = x_0, x_1, \dots, x_n$ in H , we consider the *minimum height* of P to be the smallest net length of an initial subwalk x_0, x_1, \dots, x_i , and the *maximum height* of P to be the greatest net length of an initial subwalk x_0, x_1, \dots, x_i . Note that when $i = 0$, we obtain the trivial subwalk x_0 of net length zero, and when $i = n$, we obtain the entire walk P . We shall say that P is *constricted from below* if the minimum height of P is zero

(no initial subwalk x_0, x_1, \dots, x_i has negative net length), and *constricted* if moreover the maximum height is the net length of P (no initial subwalk x_0, x_1, \dots, x_i has greater net length than x_0, x_1, \dots, x_n). We also say that P is *nearly constricted from below* if the net length of P is minus one, but all proper initial subwalks x_0, x_1, \dots, x_i with $i < n$ have non-negative net length. It is easy to see that a walk which is nearly constricted from below can be partitioned into two constricted pieces, by dividing it at any vertex achieving the maximum height.

A vertex x of H is called *extremal* if every walk starting in x is constricted from below, i.e., there is no walk starting in x with negative net length. It is clear that a balanced digraph H contains extremal vertices (we can take any x from which starts a walk with net length equal to the height of H), and an unbalanced digraph does not have extremal vertices (from any x we can find a walk of negative net length by going to an unbalanced cycle and then following it long enough in the negative direction). Moreover, in a weakly connected digraph H , any extremal vertex x is the beginning of a constricted walk of net length equal to the height of H .

For walks P from a to b , and Q from b to c , we denote by PQ the walk from a to c which is the concatenation of P and Q , and by P^{-1} the walk P traversed in the opposite direction, from b to a . We call P^{-1} the *reverse* of P . For a closed walk C , we denote by C^a the concatenation of C with itself a times.

Our main result is the following forbidden structure characterization.

Theorem 2.1 *A digraph H admits a Min-Max ordering if and only if it does not contain an induced unbalanced oriented cycle of net length greater than one, and does not contain a symmetrically invertible pair.*

An oriented cycle of H is *induced* if H contains no other arcs on the vertices of the cycle. In particular, an induced oriented cycle of length greater than one does not contain a loop. Symmetrically invertible pairs are defined below.

We define two walks $P = x_0, x_1, \dots, x_n$ and $Q = y_0, y_1, \dots, y_n$ in H to be *congruent*, if they follow the same pattern of forward and backward arcs, i.e., $x_i x_{i+1}$ is a forward (backward) arc if and only if $y_i y_{i+1}$ is a forward (backward) arc (respectively). Suppose the walks P, Q as above are congruent. We say an arc $x_i y_{i+1}$ is a *faithful arc from P to Q* , if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively), and we say an arc $y_i x_{i+1}$ is a *faithful arc from Q to P* , if it is a forward (backward) arc when $x_i x_{i+1}$ is a forward (backward) arc (respectively). We say that P, Q *avoid each other* if there is no pair of faithful arcs $x_i y_{i+1}$ from P to Q , and $y_i x_{i+1}$ from Q to P , for some $i = 0, 1, \dots, n$. A *symmetrically invertible pair*, or *sym-invertible pair*, in H is a pair of distinct vertices u, v , such that there exist congruent walks P from u to v and Q from v to u , which avoid each other.

A somewhat different notion of invertible pairs occurs in the study of list homomorphisms [13], and so we add the adjective ‘symmetrically’ or the prefix ‘sym-’ to distinguish the two concepts.

We define an auxiliary digraph H^* as follows. The vertices of H^* are all ordered pairs (x, y) of distinct vertices of H , and there is an arc in H^* from (x, y) to (x', y') just if xx', yy' are both forward arcs of H but xy', yx' are not both forward arcs of H . (Either just one is an arc, or neither is an arc). Note that in H^* we have an arc from (x, y) to (x', y') if and only if there is an arc from (y, x) to (y', x') . It follows from these definitions that a sym-invertible pair of vertices u, v in H corresponds in H^* to an oriented path between vertices (u, v) and (v, u) , i.e., H admits a sym-invertible pair if and only if there exist u, v so that (u, v) and (v, u) belong to the same weak component of H^* .

Theorem 2.2 *If H contains a sym-invertible pair, then it does not admit a Min-Max ordering.*

Proof: Indeed, suppose that $<$ is a Min-Max ordering and (x, y) and (x', y') are adjacent in H^* . Observe that if x precedes (respectively follows) x' in $<$, then y must also precede (respectively follow) y' in $<$. Hence if u, v is a sym-invertible pair in H^* , then if u is ordered before (respectively after) v , by following the avoiding congruent walks P and Q from the definition of a sym-invertible pair, we conclude that also v must be ordered before (respectively after) u . So, a sym-invertible pair implies a violation of antisymmetry, and hence it is an obstruction to the existence of a Min-Max ordering. \diamond

Theorem 2.3 *If H contains an induced unbalanced oriented cycle of net length greater than one, then it does not admit a Min-Max ordering.*

Proof: Indeed, suppose C is an induced unbalanced oriented cycle of net length $k > 1$, and let x_0 be a vertex of C in which we can start a walk P around C which is constricted from below. It is easy to see that such a vertex must exist; in fact, we may assume that even $P \setminus x_0$ is constricted from below. Then following P let x_i ($1 \leq i \leq k-1$) be the last vertex on P such that the walk from x_0 to x_i has net length i . It is easy to see that $x_i, i = 0, 1, \dots, k-1$ are all found in the first pass around C . Then $(x_0, x_1), (x_1, x_2), \dots, (x_{k-1}, x_0)$ belong to the same weak component of H^* , in violation of transitivity of $<$. Indeed, it is easy to prove, using Lemma 2.5 and the fact that C is an induced cycle, that any two pairs (x_{i-1}, x_i) and (x_i, x_{i+1}) belong to the same weak component of H^* . \diamond

Thus we shall assume that the digraph H has no induced unbalanced cycle of net length greater than one, and no sym-invertible pair.

Lemma 2.4 *Let a, b, c be three vertices of H , such that the component of H^* which contains (a, b) contains neither of $(a, c), (c, b)$.*

Let A, B, C be congruent walks starting at a, b, c respectively.

If A and B avoid each other, then B and C also avoid each other, and A and C also avoid each other.

Proof: By symmetry, it suffices to prove the claim about B and C .

Suppose $A = a_1, a_2, \dots, a_n$, $B = b_1, b_2, \dots, b_n$, and $C = c_1, c_2, \dots, c_n$ (here $a_1 = a$, $b_1 = b$, and $c_1 = c$). For a contradiction, suppose that B and C do not avoid each other, and let i be the least subscript such that both $b_i c_{i+1}$ and $c_i b_{i+1}$ are faithful arcs in H . (Note that i could be equal to $n - 1$.)

Since (a, b) and (a, c) are not in the same component of H^* , the congruent walks

$$R = a_1, \dots, a_i, a_{i+1}, a_i, \dots, a_1 \quad \text{and} \quad S = b_1, \dots, b_i, b_{i+1}, c_i, \dots, c_1$$

do not avoid each other. Since A and B do avoid each other, any faithful arcs between R and S must be between b_{i+1}, c_i, \dots, c_1 and a_{i+1}, a_i, \dots, a_1 . Suppose first there exists a subscript $j < i$ such that $a_j c_{j+1}$ and $c_j a_{j+1}$ are faithful arcs, and let j to be chosen as small as possible subject to this. Note that there is a second possibility, that $a_i b_{i+1}$ and $c_i a_{i+1}$ are the only faithful arcs. We think of this case as having $j = i$, with the understanding that c_{j+1} is replaced by b_{j+1} , and we will deal with it at the end of this proof.

Since (a, b) and (c, b) are not in the same component of H^* , the congruent walks

$$R' = a_1, \dots, a_j, a_{j+1}, c_j, \dots, c_1 \quad \text{and} \quad S' = b_1, \dots, b_j, b_{j+1}, b_j, \dots, b_1$$

do not avoid each other. Since A and B do avoid each other and since $j < i$ while i was chosen to be minimal, the faithful arcs must be $b_j a_{j+1}, c_j b_{j+1}$. Similarly, the congruent walks

$$R'' = a_1, \dots, a_j, c_{j+1}, c_j, \dots, c_1 \quad \text{and} \quad S'' = b_1, \dots, b_j, b_{j+1}, b_j, \dots, b_1$$

yield the faithful arcs $a_j b_{j+1}$ and $b_j c_{j+1}$ - contradicting the fact that A, B avoid each other.

Returning now to the special case when $j = i$, we observe that we can use the same pair of walks R', S' as above and then modify the walks

$$R'' = a_1, \dots, a_i, a_{i+1}, c_i, \dots, c_1 \quad \text{and} \quad S'' = b_1, \dots, b_i, c_{i+1}, b_i, \dots, b_1,$$

to conclude that $b_i a_{i+1}$ is again an arc, yielding the same contradiction. \diamond

Lemma 2.5 [21, 31] *Let P_1 and P_2 be two constricted walks of net length r . Then there is a constricted path P of net length r that admits a homomorphism f_1 to P_1 and a homomorphism f_2 to P_2 , such that each f_i takes the starting vertex of P to the starting vertex of P_i and the ending vertex of P to the ending vertex of P_i .*

We shall call Q a *common pre-image* of P_1 and P_2 .

We now formulate a corollary of the last two lemmas.

Corollary 2.6 *Let a, b, c be three vertices of H , such that the component of H^* which contains (a, b) contains neither of $(a, c), (c, b)$.*

Let A, B, C be three constricted walks of the same net length, starting at a, b, c respectively. Suppose that A and B are congruent and avoid each other.

Then there exists congruent common pre-images A', B', C' of A, B, C starting at a, b, c respectively, such that B' and C' avoid each other, and A' and C' also avoid each other.

Since H has no sym-invertible pairs, we conclude that if a pair (u, v) is in a weak component C of H^* , then the corresponding reversed pair (v, u) is in a different component $C' \neq C$ of H^* . Moreover, if any (x, y) also lies in C , then the corresponding reversed (y, x) must also lie in C' , since reversing all pairs on an oriented walk between (u, v) and (x, y) results in an oriented walk between (v, u) and (y, x) . Thus the components of H^* come in pairs C, C' so that the ordered pairs in C' are the reverses of the ordered pairs in C . We say the components C, C' are *dual* to each other.

3 The Algorithm

We assume that H has no induced unbalanced cycle of net length greater than one, and no sym-invertible pairs. We shall give an algorithm to construct a desired Min-Max ordering $<$. At each stage of the algorithm, some components of H^* have already been *chosen*. The chosen components define a binary relation $<$ as follows: we set $a < b$ if the pair (a, b) belongs to one of the chosen components. Whenever a component C of H^* is chosen, its dual component C' is *discarded*. The objective is to avoid a *circular chain*

$$(a_0, a_1), (a_1, a_2), \dots, (a_n, a_0)$$

of pairs belonging to the chosen components. Our algorithm always chooses a component X of maximum height from among the as yet un-chosen and un-discarded components. If X creates a circular chain, then the algorithm chooses the dual component X' . We shall show that at least one of X and X' will not create circular chain. (Note that this implies that the component X does not contain a circular chain.) Thus at the end of the algorithm we have no circular chain and hence $<$ is a total order. It is easy to see that $<$ is a Min-Max ordering. Indeed, if $i < j, s < r$ and $ir, js \in A(H)$ but $is \notin A(H)$ or $jr \notin A(H)$, then (i, j) and (r, s) are adjacent in H^* - whence we have either $i < j, r < s$ or $j < i, s < r$, contrary to what was supposed.

Theorem 3.1 *The algorithm avoids creating a circular chain.*

Thus suppose that at a certain time T there was no circular chain amongst the chosen components, that X had the maximum height from all unchosen (and undiscarded) components, and that the addition of X to the chosen components created the circular chain $(a_0, a_1), (a_1, a_2), \dots, (a_n, a_0)$, and the addition of the dual component X' created the circular chain $(b_0, b_1), (b_1, b_2), \dots, (b_m, b_0)$. We may suppose that T was minimum for which this occurs, then n was minimum value for this T , and then m was minimum value for this T and n . We may also assume that X contains the pairs $(a_n, a_0), (b_0, b_m)$, and possibly other (a_i, a_{i+1}) or (b_j, b_{j+1}) .

Let A_i be the weak component of H^* containing the pair (a_i, a_{i+1}) , and B_j be the weak component containing the pair (b_j, b_{j+1}) ; subscripts are modulo n and m respectively. (Thus $X = A_n = B'_m$.) Note that the minimality of n implies that no A_i contains a pair (a_k, a_ℓ) for subscripts (reduced modulo $n+1$) $\ell \neq k+1$ (and similarly for B_j). (This is helpful when checking the hypothesis of Lemma 2.4 and Corollary 2.6, as in Case 2 below.)

Lemma 3.2 *Suppose that none of the pairs (a_i, a_{i+1}) is extremal in its component A_i .*

Then there exists another circular chain $(a'_0, a'_1), (a'_1, a'_2), \dots, (a'_n, a'_0)$ where each (a'_i, a'_{i+1}) can be reached from the corresponding (a_i, a_{i+1}) by a walk in A_i nearly constricted from below.

Proof: Since (a_i, a_{i+1}) is not extremal, there exists a walk W_i in A_i from (a_i, a_{i+1}) to some (p_i, q_i) , which is nearly constricted from below. Corresponding to this walk in A_i , there are two walks P_i and Q_i in H , from a_i to p_i and from a_{i+1} to q_i respectively, which avoid each other. Let L_i be the maximum height of W_i (which is the same as in P_i , and Q_i).

We now explain how to choose n of the $2n$ vertices p_i, q_i which also form a circular chain. For any i , instead of a_i , we choose $a'_i = q_{i-1}$ if $L_{i-1} < L_i$, and we choose $a'_i = p_i$ otherwise. We now show that $(a'_0, a'_1), (a'_1, a'_2), \dots, (a'_n, a'_0)$ is a circular chain; it suffices to show that each (a'_i, a'_{i+1}) is in A_i .

Case 1. Suppose $L_i \leq L_{i-1}$ and $L_i \leq L_{i+1}$.

In this case, we have $a'_i = p_i, a'_{i+1} = q_i$, and (p_i, q_i) is in A_i by definition.

Case 2. Suppose $L_i \geq L_{i-1}$ and $L_i \geq L_{i+1}$.

In this case, we have $a'_i = q_{i-1}, a'_{i+1} = p_{i+1}$. We may assume that $L_{i+1} \leq L_{i-1}$ (otherwise the argument is symmetric). Consider the congruent walks $A = P_{i-1}$ from a_{i-1} to p_{i-1} and $B = Q_{i-1}$ from a_i to q_{i-1} . They are nearly constricted from below, and have maximum height L_{i-1} . Consider the following walk C from a_{i+1} to p_{i+1} : the walk C starts with a portion of Q_i , up to the maximum height L_{i-1} and then back down to a_{i+1} , followed by P_{i+1} . Note that C is also nearly constricted from below and has the same maximum height L_{i-1} . It follows that A, B, C can each be partitioned into two constricted pieces of corresponding net lengths. Since $(a_{i-1}, a_{i+1}), (a_{i+1}, a_i) \notin A_{i-1}$ by the minimality

of n , Corollary 2.6 (applied to each of the constricted pieces) implies that B and C avoid each other. Since $a'_i = q_{i-1}, a'_{i+1} = p_{i+1}$, we have a walk in H^* from (a_i, a_{i+1}) to (a'_i, a'_{i+1}) , whence $(a'_i, a'_{i+1}) \in A_i$.

Case 3. Suppose $L_{i-1} < L_i < L_{i+1}$ (or $L_{i-1} > L_i > L_{i+1}$).

In this case, we have $a'_i = q_{i-1}, a'_{i+1} = q_i$. Since the subscripts are computed modulo $n + 1$, there must exist a subscript s such that $L_s \geq L_i \geq L_{s+1}$. Now we again apply Corollary 2.6 to the walks $A = P_i, B = Q_i$, and C from a_{s+1} to p_{s+1} using P_{s+1} and a portion of Q_s , to conclude that C avoids B . Finally, we once more apply Lemma 2.4 to the three walks B, C , and D from a_i to $a'_i = q_{i-1}$ using Q_{i-1} and a portion of P_i , to conclude that D avoids B . Hence there is a walk in H^* from (a_i, a_{i+1}) to (a'_i, a'_{i+1}) , implying that $(a'_i, a'_{i+1}) \in A_i$. \diamond

This will allow us, after some additional lemmas, to complete the proof of Theorem 3.1. The details will be in the full version of the paper.

Corollary 3.3 *The following three statements are equivalent for a digraph H*

1. H admits a Min-Max ordering
2. H has no invertible pair and no induced oriented cycle of net length greater than one
3. no weak component of H^* contains a circular chain

Proof: The equivalence of (1) and (2) is Theorem 2.1. It is obvious that (1) implies (3). Finally, (3) implies (2) as an invertible pair in H is a circular chain of length two in a weak component of H^* , and the proof of Theorem 2.3 shows that an induced oriented cycle of net length greater than one yields a circular chain in a weak component of H^* . \diamond

It follows that the existence of a Min-Max ordering can be tested in polynomial time: to test (2), construct H^* , find its weak components, and test each for circular chains. Testing a weak component for circular chains amounts to looking at a set of ordered pairs, i.e., a digraph, and looking for a directed cycle. Acyclicity can be tested in linear time by topological sort.

4 Extended Min-Max Orderings

We now discuss *extended Min-Max orderings*, for digraphs H with a fixed homomorphism $\ell : H \rightarrow \vec{C}_k$. For the remainder of this discussion, the digraph H and the homomorphism ℓ is fixed. (The standard Min-Max orderings may be viewed as the special case $k = 1$.) Assume the vertices of \vec{C}_k are $0, 1, \dots, k - 1$, and let $V_i = \ell^{-1}(i)$. A k -Min-Max ordering of H is a linear ordering $<$ of each V_i , so that the Min-Max condition ($i < j, s < r$ and

$ir, js \in A(H)$ imply $is \in A(H)$ and $jr \in A(H)$) is satisfied for i, j and s, r in any two circularly consecutive sets V_i and V_{i+1} respectively (subscript addition modulo k). Note that a Min-Max ordering is also a k -Min-Max ordering for any k and ℓ ; however, there are digraphs with a k -Min-Max ordering that do not have a Min-Max ordering - for instance \vec{C}_k .

We shall consider a subgraph of H^* defined as follows. The digraph $H^{(k)}$ is the subgraph of H^* induced by all ordered pairs (x, y) of with $\ell(x) = \ell(y)$. We say that (u, v) is a *symmetric k -invertible pair* (or a *sym- k -invertible pair*) in H if there is in $H^{(k)}$ an oriented walk joining (u, v) and (v, u) . Note that each sym- k -invertible pair is just a sym-invertible pair in H in which u and v have $\ell(u) = \ell(v)$. Note that H may contain sym-invertible pairs, but none with $\ell(u) = \ell(v)$. Consider, for instance the directed hexagon \vec{C}_6 on $0, 1, 2, 3, 4, 5$. The pair $0, 3$ is sym-invertible, but not sym-6-invertible. Note also that there is a homomorphism ℓ of \vec{C}_6 to \vec{C}_3 in which $\ell(0) = \ell(3)$, in which the pair $0, 3$ is 3-invertible.

The extended version of our main theorem is the following.

Theorem 4.1 *A digraph H with a homomorphism ℓ to \vec{C}_k admits a k -Min-Max ordering if and only if it does not contain an induced unbalanced oriented cycle of net length other than k , and does not contain a sym- k -invertible pair.*

Proof: The proof is similar to the case $k = 1$, Theorem 2.3. The only additional twist occurs in the case when X is unbalanced, where we need to observe that each part V_i must contain a vertex from which there is some walk of infinite height which is constricted from below. \diamond

Theorem 4.1 can be used to prove the following result conjectured in [18].

Theorem 4.2 *If H has a homomorphism to some \vec{C}_k which admits a k -Min-Max ordering then $\text{MinHOM}(H)$ is polynomial time solvable. Otherwise, $\text{MinHOM}(H)$ is NP-complete.*

The positive direction (the existence of a k -Min-Max ordering implies a polynomial time algorithm) is proved in [18]. We prove Theorem 4.2 using our characterization in Theorem 4.1, by showing that $\text{MinHOM}(H)$ is NP-complete if H contains a sym- k -invertible pair or an induced unbalanced oriented cycle of net length other than k ; this is done in the appendix.

References

- [1] R.S. Takhanov, Dichotomy theorem for general minimum cost homomorphism problem, arXiv:0708.326v5.

- [2] V.E. Alekseev and V.V. Lozin, Independent sets of maximum weight in (p, q) -colorable graphs, *Discrete Mathematics* 265 (2003) 351–356.
- [3] J. Bang-Jensen, P. Hell, and G. MacGillivray, Hereditarily hard H-colouring problems, *Discrete Math.* 138 (1995) 75–92.
- [4] L. Barto, M. Kozik, and T. Niven, The CSP dichotomy holds for digraphs with no sources and sinks (a positive answer to a conjecture of Bang-Jensen and Hell), *SIAM J. Computing*, in print.
- [5] L. Barto, M. Kozik, M. Maróti, and T. Niven, CSP dichotomy for special triads, *Proc. AMS*, in print.
- [6] A. A. Bulatov, Tractable conservative constraint satisfaction problems, 18th IEEE SOLCS (2003) pp. 321–330.
- [7] A. A. Bulatov, P. G. Jeavons, and A. A. Krokhin, Constraint satisfaction problems and finite algebras, 27th ICALP (2000) pp. 272–282.
- [8] C Carvalho, V. Dalmau, A. Krokhin, Caterpillar duality for constraint satisfaction problems, 23rd Annual IEEE Symposium LICS, 307-316.
- [9] D. Cohen, M. C. Cooper, P. Jeavons, Constraints, consistency and closure, *Artificial Intelligence* 101 (1998), 251-265.
- [10] D. Cohen, M. Cooper, P. Jeavons, and A. Krokhin, A maximal tractable class of soft constraints. *J. Artif. Intell. Res.* 22 (2004), 1–22.
- [11] T. Färnqvist and P. Jonsson, Bounded treewidth and CSP-related problems, ISAAC2007.
- [12] T. Feder, P. Hell, and J. Huang, Bi-arc graphs and the complexity of list homomorphisms, *J. Graph Theory* 42 (2003) 61 - 80.
- [13] T. Feder, P. Hell, J. Huang, and A. Rafiey, Adjusted interval digraphs, *Electronic Notes in Discrete Mathematics* 32 (2009) 83–91.
- [14] T. Feder, M. Vardi, The computational structure of monotone monadic SNP and constraint satisfaction: a study through datalog and group theory, *SIAM Journal on Computing* 28 (1998) 57–104.
- [15] A. Gupta, P. Hell, M. Karimi, and A. Rafiey, Minimum cost homomorphisms to reflexive digraphs, LATIN 2008.
- [16] G. Gutin, P. Hell, A. Rafiey and A. Yeo, A dichotomy for minimum cost graph homomorphisms, *European J. Combin.* 29 (2008) 900–911.

- [17] G. Gutin, A. Rafiey and A. Yeo, Minimum cost and list homomorphisms to semicomplete digraphs, *Discrete Appl. Math.* 154 (2006), 890–897.
- [18] G. Gutin, A. Rafiey and A. Yeo, Minimum Cost Homomorphisms to Semicomplete Bipartite Digraphs, *SIAM Journal of Discrete Math.* 22 (2008) 1624–1639.
- [19] G.Gutin, A. Rafiey, A.Yeo, Minimum Cost Homomorphism to Oriented Cycles. Submitted.
- [20] G. Gutin, A. Rafiey, A. Yeo and M. Tso, Level of repair analysis and minimum cost homomorphisms of graphs. *Discrete Appl. Math.* 154 (2006), 881–889.
- [21] R. Haggkvist, P. Hell, D. J. Miller, and V. Neumann Lara, On multiplicative graphs and the product conjecture, *Combinatorica* 8 (1988) 7181.
- [22] M. M. Halldorsson, G. Kortsarz, and H. Shachnai, Minimizing average completion of dedicated tasks and interval graphs. Approximation, Randomization, and Combinatorial Optimization (Berkeley, Calif, 2001), Lecture Notes in Computer Science, vol. 2129, Springer, Berlin, 2001, pp. 114–126.
- [23] P. Hell and J. Nešetřil, On the complexity of H -colouring, *J. Combin. Theory B* 48 (1990), 92–110.
- [24] P. Hell and J. Nešetřil, *Graphs and Homomorphisms*. Oxford University Press, Oxford, 2004.
- [25] K. Jansen, Approximation results for the optimum cost chromatic partition problem. *J. Algorithms* 34 (2000) 54–89.
- [26] P. G. Jeavons, On the algebraic structure of combinatorial problems, *Theoret. Comput. Sci.* 200 (1998) 185–204.
- [27] T. Jiang and D. B. West, Coloring of trees with minimum sum of colors. *J. Graph Theory* 32 (1999), 354–358.
- [28] L.G. Kroon, A. Sen, H. Deng, and A. Roy, The optimal cost chromatic partition problem for trees and interval graphs, Graph-Theoretic Concepts in Computer Science (Cadenabbia, 1996), Lecture Notes in Computer Science, vol. 1197, Springer, Berlin, 1997, pp. 279–292.
- [29] J. Spinrad, **Efficient Graph Representations**, Fields Institute Monographs, AMS 2003.
- [30] K. Supowit, Finding a maximum planar subset of a set of nets in a channel. *IEEE Trans. Computer-Aided Design* 6 (1987), 93–94.
- [31] X. Zhu, A simple proof of the multiplicatively of directed cycles of prime power length. *Discrete Appl. Math.* , 36 (1992) 333–345.

5 Appendix

5.1 Proof of Theorem 3.1

We distinguish two principal cases, depending on whether or not the component X is balanced.

We first assume that the component X is balanced.

Suppose the height of X is h .

Lemma 5.1 *Suppose some (a_k, a_{k+1}) is extremal in A_k .*

Let $(a_i, a_{i+1}), (a_j, a_{j+1})$ be distinct non-extremal pairs in A_i, A_j respectively, and let W_i, W_j be walks in A_i, A_j starting from $(a_i, a_{i+1}), (a_j, a_{j+1})$ respectively, that are nearly constricted from below. Let L_i, L_j be the maximum heights of W_i, W_j respectively.

Then $L_i > h$ or $L_j > h$.

Proof: Suppose $L_i \leq h, L_j \leq h$, and assume, without loss of generality, that $L_i \leq L_j$. Since some (a_k, a_{k+1}) is extremal, we may assume that neither (a_{i-1}, a_i) , nor (a_{j+1}, a_{j+2}) initiate walks of negative net length with maximum height at most h . Thus each of $(a_{i-1}, a_i), (a_{j+1}, a_{j+2})$ is either extremal, and thus initiate a constricted walk of net length h , or initiates a walk of negative net length, with maximum height greater than h , and hence again initiates a constricted walk of net length h . Thus we have

- a constricted walk U_i of net length h from a_i
- a walk V_i , nearly constricted from below, from a_i to some p
- a constricted walk U_{j+1} of net length h from a_{j+1}
- a constricted walk U_{j+2} of net length h from a_{j+2} , which avoids U_{j+1} and is congruent to it
- a walk V_j , nearly constricted from below, from a_j , and
- a walk V_{j+1} , nearly constricted from below, from a_{j+1} to some q , which avoids V_j and is congruent to it.

Consider the three walks A, B, C , where A is the reverse of V_{j+1} (starting in q), B is the reverse of V_j , and C is the reverse of V_i followed by a suitable piece of U_i (and its reverse) as needed to have the same maximum height L_j as V_j . Each of these walks consists of two constricted pieces and hence we can apply Corollary 2.6 twice to conclude that there exist congruent pre-images A' and C' of A and C respectively, which avoid

each other. We can also apply Corollary 2.6 to the constricted walks U_{j+1}, U_{j+2}, U_i to conclude that there are congruent pre-images A'', C'' of U_{j+1}, U_j respectively, which avoid each other. Concatenating A' with A'' and C' with C'' , we conclude that (p, q) belongs to a component of H^* which has height greater than h ; this means that before X we should have chosen the component of H^* containing (a_i, a_j) , which is a contradiction. \diamond

Lemma 5.2 *If any (a_i, a_{i+1}) is extremal in A_i , then (a_n, a_0) is extremal in $X = A_n$.*

Proof: Suppose (a_n, a_0) is not extremal. By Lemma 5.1, it remains to consider the case when both (a_0, a_1) and (a_{n-1}, a_n) are extremal. Since (a_0, a_1) is extremal, there exists a constricted walk in H^* starting from (a_0, a_1) of net length equal to the height of A_0 , which is at least h , according to our algorithm. Similarly, there exists a constricted walk from (a_{n-1}, a_n) of net length equal to the height of A_{n-1} , which is also at least h . From the walk in A_{n-1} , we extract a constricted walk A starting in a_{n-1} , and a congruent constricted walk B starting in a_n such that A, B have net length h and avoid each other. From the walk in A_0 we moreover extract a walk C starting in a_0 which is also constricted and has net length h . Now Corollary 2.6 ensures that B and C have congruent pre-images B' and C' which avoid each other. Let B'', C'' be two congruent walks of negative net length from a_n, a_0 respectively, which avoid each other; such walks exist since (a_n, a_0) is not extremal. Now taking the concatenations of $(B'')^{-1}$ with B' and $(C'')^{-1}$ with C' yields a walk in X of net length greater than h , which is a contradiction. \diamond

Thus Lemma 3.2 ensures that we may assume that (a_n, a_0) is extremal in X (and similarly for (b_0, b_m)). The proof now distinguishes whether or not X contains another pair (a_i, a_{i+1}) (or similarly for (b_j, b_{j+1})).

Suppose first that some $(a_i, a_{i+1}) \in X$, and let W be a walk from (a_n, a_0) to (a_i, a_{i+1}) in X . We observe that the net length of W must be zero. Indeed, since (a_n, a_0) is extremal in X , the net length of W must be non-negative. If the net length were positive, then W^{-1} would be a walk from (a_i, a_{i+1}) of negative net length and with maximum height less than h . Thus Lemma 5.1 implies that both $(a_{i-1}, a_i), (a_{i+1}, a_{i+2})$ initiate walks of net length h , yielding walks $U_{i-1}, U_i, U_{i+1}, U_{i+2}$ of net length h , from $U_{i-1}, U_i, U_{i+1}, U_{i+2}$, respectively. Here U_{i-1}, U_i are congruent constricted walks that avoid each other, and hence Corollary 2.6 implies that there are pre-images of U_i, U_{i+1} of net length h that are congruent and avoid each other. This yields a walk in X from (a_i, a_{i+1}) of net length h - and concatenated with W we obtain a walk in X from (a_n, a_0) of net length strictly greater than h , which is impossible.

Thus the net length of W is zero, and hence it can be partitioned into two constricted pieces, U from (a_n, a_0) to some vertex (z_1, z_2) of maximum height, and V from (z_1, z_2) to (a_i, a_{i+1}) . Let U_1 (respectively U_2) denote the corresponding walk from a_n to z_1 (respectively from a_0 to z_2), and similarly for V_1, V_2 . Then Lemma 2.4 applied to U_1, U_2, V_2 implies that (z_1, z_2) and (a_n, a_{i+1}) are in the same component of H^* ; however, $(z_1, z_2) \in X$, so $(a_n, a_{i+1}) \in X$, contrary to the minimality of n .

Thus we conclude that X does not contain another (a_i, a_{i+1}) or (b_j, b_{j+1}) . In other words, before time T we have chosen all the pairs

$$(a_0, a_1), \dots, (a_{n-1}, a_n), (b_0, b_1), \dots, (b_{m-1}, b_m),$$

and then at time T we chose the component X containing (a_n, a_0) as well as (b_0, b_m) . Consider a fixed walk W in X from (a_n, a_0) to (b_0, b_m) . Since (a_n, a_0) , and by symmetry also (b_0, b_m) , is extremal, W must have net length zero. Moreover, we may assume that W reaches some vertex (z_1, z_2) of maximum height h . Thus W consists of two constricted walks U, V . Let again U_1 (respectively U_2) be the corresponding walk in H from a_n (respectively from a_0) to a vertex of maximum height, and similarly let V_1 (respectively V_2) be the corresponding walks from the vertices of maximum height to b_0 (respectively b_m).

We shall prove first that there is a constricted walk of net length h from a_1 . Indeed, the component A_0 , containing the vertex (a_0, a_1) must have height at least h , according to the rules of our algorithm. If (a_0, a_1) does not initiate a walk of net length h , it must not be extremal, i.e., it must initiate a walk of negative net length. The same argument yields a walk of negative net length from (a_1, a_2) . Since such walks contain walks that are nearly constricted from below, we obtain a contradiction with Lemma 5.1. A similar argument applies to b_1 .

Thus there are constricted walks of net length h from both a_1 and b_1 , say R and S respectively. We can now use Corollary 2.6 on the walks $A = U_1, B = U_2, C = R$, and again on the walks $A = V_1, B = V_2, C = R^{-1}$ to deduce that U_2 concatenated with V_2 and R concatenated with R^{-1} avoid each other, whence (a_0, a_1) and (b_m, a_1) are in the same component of H^* . By a similar argument we also deduce that (b_0, b_1) and (a_n, b_1) are also in the same component of H^* . This is impossible, as it would mean that at time $T-1$ there already was a circular chain, namely $(b_m, a_1), (a_1, a_2), \dots, (a_n, b_1), (b_1, b_2), \dots, (b_{m-1}, b_m)$.

This completes the proof of Theorem 3.1 in case X is balanced.

We now assume the component X is unbalanced.

In this case, the rules of the algorithm imply that each component A_i and B_j is also unbalanced. Thus each of the components contains an oriented cycle of net length one, and hence there is a closed walk of net length one, or minus one, starting in any vertex in any of these components. In particular, as we observed before, an unbalanced digraph does not contain any extremal vertices. We shall define a vertex u in an unbalanced digraph to be *weakly extremal* if there is a walk starting from u which is constrained from below and has infinite maximum height. Each oriented cycle of positive net length, and hence each unbalanced digraph, contains a weakly extremal vertex.

Recall our assumptions that X contains $(a_n, a_0), (b_0, b_m)$ and maybe other pairs, creating the circular chain $(a_0, a_1), (a_1, a_2), \dots, (a_n, a_0)$ in X and the circular chain $(b_0, b_1), (b_1, b_2), \dots, (b_m, b_0)$ in X' .

We first claim that we may assume that each (a_i, a_{i+1}) (and similarly each (b_j, b_{j+1})) is weakly extremal. Indeed, suppose there is a walk in A_i from (a_i, a_{i+1}) to some weakly extremal vertex (e_i, e_{i+1}) , of negative net length ℓ_i . Let ℓ be the minimum of all $\ell_i, i = 0, \dots, n$. Since (a_i, a_{i+1}) initiates a closed walk in A_i of net length minus one, there is a walk from each (a_i, a_{i+1}) to the weakly extremal vertex (e_i, e_{i+1}) of net length ℓ . Now we apply Lemma 3.2 ℓ times to obtain a circular chain $(a'_0, a'_1), (a'_1, a'_2), \dots, (a'_n, a'_0)$. It follows from the proof of Lemma 3.2 that each (a'_i, a'_{i+1}) has a walk of net length zero to (e_i, e_{i+1}) ; this means that each (a'_i, a'_{i+1}) is weakly extremal.

As in the balanced case, we first assume that some $(a_i, a_{i+1}) \in X$. Then there is a walk W from (a_n, a_0) to (a_i, a_{i+1}) in X of net length zero. Indeed, the argument above shows that both (a_n, a_0) and (a_i, a_{i+1}) have a walk of net length ℓ to (e_i, e_{i+1}) , since in this case $A_i = A_n = X$. As before, X can be partitioned into two constricted pieces, U and V , and Lemma 2.4 implies that $(a_n, a_{i+1}) \in X$, contrary to the minimality of n .

If X does not contain another (a_i, a_{i+1}) or (b_j, b_{j+1}) , we again proceed as in the balanced case. There exists a walk W in X of net length zero from (a_n, a_0) to (b_0, b_m) . (Both (a_n, a_0) and (b_0, b_m) can reach (e_n, e_0) with walks of the same net length.) Let L be the maximum height of W . Thus W consists of two constricted walks U, V . Let again U_1 (respectively U_2) be the corresponding walk in H from a_n (respectively from a_0) to a vertex of maximum height, and similarly let V_1 (respectively V_2) be the corresponding walks from the vertices of maximum height to b_0 (respectively b_m). Since (a_0, a_1) is weakly extremal, there is a constricted walk of net length L from a_1 , and for a similar reason, there is such a walk also from b_1 .

We can now use Corollary 2.6 as in the balanced case, to deduce that (a_0, a_1) and (b_m, a_1) are in the same component of H^* , and that (b_0, b_1) and (a_n, b_1) are in the same component of H^* , yielding the same contradiction.

This completes the proof of Theorem 3.1.

5.2 The NP-completeness Claims

Our basic NP-completeness tool is summarized in the next lemma.

Lemma 5.3 *Let H be a digraph and x, y two vertices of H ; let S be a digraph and s, t two vertices of S . Suppose we have costs $c_j(i)$ of mapping vertices i of S to vertices j of H where $c_x(s) = c_x(t) = 1, c_y(s) = c_y(t) = 0$, and such that there exists*

- *a homomorphism $f : S \rightarrow H$ mapping s to x and t to y of total cost 1 (i.e., in which all other vertices of S , different from s, t , map to vertices of H with costs zero)*
- *a homomorphism $g : S \rightarrow H$ mapping s to x and t to x of total cost 2 (other vertices map with costs zero)*

- a homomorphism $h : S \rightarrow H$ mapping s to y and t to x , of total cost 1 (other vertices map with costs zero)
- but no homomorphism $S \rightarrow H$ mapping s to y and t to y of cost at most $|V(S)|$.

Then $\text{MinHOM}(H)$ is NP-complete.

Proof: Let G be an arbitrary graph, an instance of the maximum independent set problem. We construct a corresponding instance D of $\text{MinHOM}(H)$ by replacing every edge of G by a copy of S . Note that D contains all *old* vertices of G , as well as the *new* vertices each lying in a separate copy of S . The costs $c_i(j), i \in V(H), j \in V(D)$, are defined as follows.

- if v is an old vertex of G , then $c_x(v) = 1, c_y(v) = 0$, and $c_z(v) = |V(G)|$ for all other $z \in V(H)$,
- if v is a new vertex of D lying in a copy of S , its costs are determined by the corresponding costs $c_j(v)$ in S .

Note that since we have $c_x(s) = c_x(t) = 1, c_y(s) = c_y(t) = 0$, the two parts of the definition don't conflict. We now claim that G has an independent set of size k if and only if there exists a homomorphism of D to H of cost $|V(G)| - k$. Indeed, if I is an independent set in G , we define a homomorphism $\phi : D \rightarrow H$ by setting $\phi(j) = y$ if $j \in I$, $\phi(j) = x$ if $j \in V(G) \setminus I$, and extending this mapping to a homomorphism of D to H , using the mappings f, g, h . It is clear that the cost of ϕ is exactly $|V(G)| - |I|$. Conversely, let f be any homomorphism of D to H of total cost less than $|V(G)|$. Thus the old vertices of G must map to either x or y . If two adjacent vertices are mapped to y we incur a cost of at least $|V(S)| \geq 2$. Thus we may assume that those vertices that map to y are independent. Since the old vertices of G that map to x contribute a cost of one each, we conclude that if there is a homomorphism of cost $|V(G)| - k$ then there is an independent set of size k in G . \diamond

One example in which we can easily use this lemma deals with a special case of sym-invertible pairs.

Corollary 5.4 *Suppose u, v is a sym-invertible pair in H with corresponding walks P, Q , such that there are some faithful arcs from P to Q but there are no faithful arcs from Q to P .*

Then the problem $\text{MinHOM}(H)$ is NP-complete.

Proof: We assume $P = u = a_1 \dots a_n = v, Q = v = b_1 \dots b_n = u$, and let $S = s_1 \dots s_n$ be a *path* (all vertices are distinct) congruent to P (and Q). Define the cost of mapping

vertices of S to H as follows. If $c_u(s_1) = c_u(s_n) = 1$, and $c_v(s_1) = c_v(s_n) = 0$, and $c_{a_i}(s_i) = c_{b_i}(s_i) = 0$ for $1 < i < n$. In any other case the cost is n .

Clearly there are obvious homomorphisms $\phi : S \rightarrow P$ and $\psi : S \rightarrow Q$. Define also $\zeta : S \rightarrow H$ to be the homomorphism defined by $\zeta(s_i) = a_i$ for $1 \leq i \leq k$ and $\zeta(s_i) = b_i$ for $k+1 \leq i \leq n$. Let $a_t b_{t+1}$ be a faithful arc from P to Q . Suppose there is homomorphism $g : V(S) \rightarrow V(P) \cup V(Q)$ such that $g(s_1) = g(s_n) = v$. Then the cost of g is at least n unless $g(r_i)$ is a_i or b_i . Since $g(s_1) = g(s_n) = v$, there has to be a faithful arc from Q to P in H , which is a contradiction. Now by apply Lemma 5.3 for P, Q and S $\text{MinHOM}(H)$. \diamond

We next consider the case where some sym-invertible pair has faithful arcs both from P to Q and from Q to P .

It was noted in [16] (using [2]) that the following problem Π_3 is NP-complete. Given a three-coloured graph G and an integer k , decide if there exists an independent set of k vertices. It is easy to see that this fact can be generalized to the following problem

Π_{2m+1} :

Given a graph G with a homomorphism $f : G \rightarrow C_{2m+1}$, decide if there exists an independent set of k vertices.

Lemma 5.5 *Each problem Π_{2m+1} is NP-complete.*

Proof: Modify every instance G of Π_{2m-1} to an instance G' of Π_{2m+1} by replacing each edge of G between classes $f^{-1}(1)$ and $f^{-1}(2)$ by a path of length three. \diamond

We apply this result as follows.

Lemma 5.6 *Suppose u, v is a sym-invertible pair in H with corresponding walks P, Q , such that there are faithful arcs from P to Q as well as faithful arcs from Q to P .*

Then $\text{MinHOM}(H)$ is NP-complete.

Proof: The walks $P = x_0, x_1, \dots, x_n$ and $Q = y_0, y_1, \dots, y_n$ can be organized into segments $P_1, \dots, P_k, Q_1, \dots, Q_k$, where for each i all faithful arcs between P and Q go from P to Q or from Q to P . Assume $P_i = x_{r_{i-1}}, x_{r_{i-1}+1}, \dots, x_{r_i}$ and $Q_i = y_{r_{i-1}}, y_{r_{i-1}+1}, \dots, y_{r_i}$ with $r_0 = 0, r_k = n$, and assume, without loss of generality, that there are faithful arcs from P_1 to Q_1 but no faithful arcs from Q_1 to P_1 , there are faithful arcs from Q_2 to P_2 but no faithful arcs from P_2 to Q_2 , etc. Note that if k is odd, the faithful arcs of the last segment go from Q to P , and if k is even, they go from P to Q . Let R_i be a path congruent to P_i (and Q_i), and for simplicity assume that $R_i = r_{i-1}, \dots, r_i$.

Case 1. Assume k is odd.

We reduce Π_k to $\text{MinHOM}(H)$ as follows. Consider an instance of Π_k , namely, a graph G with a homomorphism f to C_k . Suppose the vertices of C_k are $1, 2, \dots, k$ (consecutively,

and viewed modulo k). Replace each edge uv of G with $u \in f^{-1}(i)$ and $v \in f^{-1}(i+1)$ (modulo k) by a copy $R_i(u, v)$ of R_i , identifying r_{i-1} with u and r_i with v , obtaining a digraph D . The costs of mapping an old vertex (from G) u in $f^{-1}(i)$ with i odd will be $c_{x_{r_i}}(u) = 1, c_{y_{r_i}}(u) = 0$, while the costs of mapping an old vertex u in $f^{-1}(i)$ with i even will be $c_{x_{r_i}}(u) = 0, c_{y_{r_i}}(u) = 1$. For vertices inside the substituted copies of R , we proceed as above, defining their costs to be zero only for the corresponding vertices in $R(u, v)$. All other costs are $|V(G)|$.

Suppose i is odd. Each homomorphism of R_i to D taking r_{i-1} to $x_{r_{i-1}}$ and r_i to y_{r_i} has a very high cost, but all other possibilities (r_{i-1} to $x_{r_{i-1}}$ and r_i to x_{r_i} ; r_{i-1} to $y_{r_{i-1}}$ and r_i to y_{r_i} ; and r_{i-1} to $y_{r_{i-1}}$ and r_i to x_{r_i}) have cost 1. A similar analysis applies to i even. A special consideration is needed for the last segment R_k , where we use the fact that $x_{r_k} = x_n = y_0$ and $y_{r_k} = y_n = x_0$.

As in the proof of Corollary 5.4, these facts imply that G has an independent set of size ℓ if and only if D has a homomorphism to H of cost $|V(G)| - \ell$.

Case 2. Assume k is even.

In this case instead of the sym-invertible pair u, v with walks P, Q we consider the sym-invertible pair y_{r_1}, x_{r_1} with walks P', Q' where $P' = y_{r_1}, \dots, y_{r_2}, \dots, y_{r_{k-1}}, \dots, y_{r_k} = y_n = x_0, \dots, x_{r_1}$, and $Q' = x_{r_1}, \dots, x_{r_2}, \dots, x_{r_{k-1}}, \dots, x_{r_k} = x_n = y_0, \dots, y_{r_1}$. Note that there are no faithful arcs from $x_{r_{k-1}}, \dots, x_{r_k} = x_n = y_0, \dots, y_{r_1}$ to $y_{r_{k-1}}, \dots, y_{r_k} = y_n = x_0, \dots, x_{r_1}$. Thus we obtain an odd number of segments and we can proceed as above, unless $k = 2$ in which case we only have one segment and Corollary 5.4 applies. \diamond

We can now handle the case when H is balanced. Recall that this means that the vertices of H have levels $0, 1, \dots, h$ so that each arc goes from some level i to level $i+1$. It is easy to see that in a balanced digraph a sym-invertible pair u, v must have u and v on the same level. Thus all sym- k -invertible pairs have $k = 1$, i.e., we only have sym-invertible pairs. Therefore, the NP-completeness part of Theorem 4.2 in this case reduces to the following.

Theorem 5.7 *If a balanced digraph H contains a sym-invertible pair, then $\text{MinHOM}(H)$ is NP-complete.*

Proof: By Corollary 5.4 and Lemma 5.6, we may assume that we have a sym-invertible pair u, v and corresponding walks P, Q with no faithful arcs between P and Q . Consider the walk W in H^* from (u, v) to (v, u) corresponding to P and Q . If some (a, b) lies on W , then there is a walk in H^* from (a, b) to (b, a) (because H^* has an arc from (x, y) to (x', y') if and only if it has an arc from (y, x) to (y', x')). Thus we may assume that u, v are on the lowest level of P and Q . Let z be vertex on the highest level of P and let w be the corresponding vertex on Q . Let R be the walk obtained by following Q from v to w and then following Q^{-1} back from w to v . Let the path S be the common pre-image of P, Q ,

and R , obtained by applying Lemma 2.5 twice, since P, Q, R consist of two constricted pieces. Let f be the corresponding homomorphism of S to P , let g be the corresponding homomorphism of S to Q , and let h be the corresponding homomorphism of S to R . We define the cost of mapping an internal vertex j of S to a vertex i of H to be zero if $i \in \{f(j), g(j), h(j)\}$; the cost of mapping the first and the last vertex of S to v is 1 and to u is zero. In all other cases the cost is $|V(S)|$. Note that there is no homomorphism from S to H which maps both beginning and end of S to u of total cost smaller than $|V(S)|$, as otherwise there would be a faithful arc from P to Q . Now by applying Lemma 5.3 to S and f, g, h we conclude that $\text{MinHOM}(H)$ is NP-complete. \diamond

Corollary 5.8 *Theorem 4.2 holds for balanced digraphs H .*

Specifically, for a balanced digraph H the problem $\text{MinHOM}(H)$ is polynomial time solvable if H has a Min-Max ordering, and is NP-complete otherwise.

We observe that the same proof applies even in unbalanced digraphs H as long as P (and hence Q) has net length zero. Specifically, if *any* digraph H has an invertible pair u, v with corresponding walks P, Q which have net length zero, then $\text{MinHOM}(H)$ is NP-complete.

Thus we may now focus on unbalanced digraphs H .

Theorem 5.9 *Suppose H is weakly connected and contains two induced oriented cycles C_1, C_2 , with net lengths $k, n > 0, k \neq n$.*

Then $\text{MinHOM}(H)$ is NP-complete.

We will use the following analogue of Lemma 2.5 for infinite walks which are constricted in the infinite sense, i.e., are constricted from below and have infinite height.

Corollary 5.10 *Let P_1 and P_2 be two walks of infinite height, constricted from below. Assume that P_i starts in $p_i, i = 1, 2$, and let q_i be a vertex on P_i , such that the infinite portion of P_i starting from q_i is also constricted from below, and the portions of P_i from p_i to q_i have the same net length, for $i = 1, 2$.*

Then there is an oriented path P that admits homomorphisms f_i to P_i taking the starting vertex of P to p_i and the ending vertex of P to q_i , for $i = 1, 2$.

Proof of the Corollary: Let P'_i be the portion of P_i from p_i to q_i , and suppose, without loss of generality, that the height h of P'_1 is greater than or equal to the height of P'_2 . Let r_i be the first vertex after q_i . (or equal to q_i) on P_i , such that the net length from p_i to r_i is h . Let R_i be the subwalk of P_i from p_i to r_i . Now Lemma 2.5 implies that there is a path R with homomorphisms f_i to R_i taking the beginning of R to p_i and

the end of R to r_i . Suppose x is the last vertex on P'_1 with $f_1(x) = q_1$: if $f_2(x) = q_2$, we are done, so suppose $f_2(x) = y \neq q_2$. Now consider the subwalk Y of P'_2 joining y and q_2 : it has net length zero and is constricted from below, because the portion of R between x and the end of R has net length zero and is constricted from below. Let h' be the height of Y , and let X be the walk on P'_1 from q_1 to the first vertex making a net length h' and then back to q_1 . Since X and Y have the same height and have net length zero, we can split them into two constricted pieces, and so Lemma 2.5 implies that there is a path R' which is a common pre-image of X and Y . Concatenating R with R' yields a path P and we can extend the homomorphisms f_i to P so that also the ending vertex of P is taken to q_i , for $i = 1, 2$. \diamond

Proof of the Theorem: Suppose $k > n$, so k does not divide n . We may assume that H is minimal, in the sense that no weakly connected subgraph H' of H with fewer vertices contains two induced cycles with different non-zero net lengths. Indeed, if H' were such a subgraph, then $\text{MinHOM}(H')$ would be polynomially reduced to $\text{MinHOM}(H)$ by the cost of mapping to vertices of H not in H' very high.

Each cycle $C_i, i = 1, 2$, contains a vertex u_i such that the walk starting in u_i and following C_i (in the positive direction) is constricted from below. Let U be a walk in H from u_1 to u_2 , and let u be a vertex on U of minimum height. By minimality, we may assume $V(H) = V(C_1) \cup V(C_2) \cup U$. Let $P_i, i = 1, 2$, be the walk from u to u_i following U (or U^{-1}), then once around C_i (in the positive direction), and then back from u following U^{-1} (or U). It follows that each P_i is constricted from below. The net length of P_1 is k and the net length of P_2 is n . Let $Q_i, i = 1, 2$, be the infinite walk starting at u obtained by repeatedly concatenating P_i , and let Q'_i be the two-way infinite walk obtained by expanding Q_i in the opposite direction by repeatedly concatenating P_i^{-1} .

Let d be greatest common divisor of n and k , and let $a = k/d - 2$. Thus $(a + 2)n$ is the smallest positive common multiple of n and k . We now define the following three walks W_1, W_2, W_3 in H of net length $(a + 1)n$.

1. The walk W_1 starts at u and follows Q_1 going around P_1 until the last vertex v such that the net length of the resulting walk is $(a + 1)n$
2. W_2 also starts at u and follows Q_2 going around P_2 fully $(a + 1)$ times, ending at u
3. W_3 starts at v and follows P_1 until the first occurrence of u , and then continues a times around P_2 , ending again at u .

Now we define, in analogy with Q_1, Q_2 , also the infinite walk Q_3 , obtained from W_3 by continuing to go around P_2 . Because we chose v to be the last on Q_1 with the right net length, the walk W_3 is constricted from below; of course W_1, W_2 are also constricted from below. Hence Q_1, Q_2, Q_3 are also constricted from below; they have infinite heights because C_1, C_2 have positive net length. Thus we can apply Corollary 5.10 to Q_1, Q_2, Q_3 , obtaining

a common pre-image which is a path S , say $s = s_0, s_1, \dots, s_q = t$, with homomorphisms f, g, h of S to Q_1, Q_2, Q_3 respectively, such that

1. $f(s) = u, f(t) = v$
2. $g(s) = g(t) = u$
3. $h(s) = v, h(t) = u$

Note that the walk W_1' equal to $u = f(s_0), f(s_1), \dots, f(s_q) = v$, the walk W_2' equal to $v = g(s_0), g(s_1), \dots, g(s_q) = u$, and the walk W_3' equal to $v = h(s_0), h(s_1), \dots, h(s_q) = u$ are congruent.

Assume first that W_1', W_3' do not avoid each other, i.e., for some i we have both the faithful arcs (forward or backward) $f(s_i)h(s_{i+1}), h(s_i)f(s_{i+1})$. Note that $W_1' \cup W_2' \cup W_3'$ contains all the vertices of H , so the minimality of H easily implies that all four vertices $f(s_i), h(s_i), f(s_{i+1}), h(s_{i+1})$ must belong to $C_1 \cup C_2$. Since the cycles are induced. we must have two vertices in each cycle. Up to symmetry, we may assume we have forward arcs $ab \in C_1$ and $cd \in C_2$, as well as forward arcs ad, cb in H . Then, say, $a = f(s_i)$, $b = f(s_{i+1})$, $c = h(s_i)$, $d = h(s_{i+1})$.

We first claim that C_1, C_2 do not have common vertices, or arcs joining them other than ad, cb . Otherwise, let x on C_1 be the first vertex following b in the direction opposite to a , equal to or adjacent with some y on C_2 , and assume that y is the first vertex of C_2 following d , in the direction opposite to c , adjacent to x . Consider the cycle D_1 with arcs ab, ad, xy , the portion of C_1 between b and x not containing a , and the portion of C_2 between d and y not containing c , and the cycle D_2 with arcs cb, cd, xy , and the same portions of C_1, C_2 . The cycles D_1, D_2 have the same net length m . If m is not zero and not k , we could delete c and obtain a smaller weakly connected H' with two different non-zero net lengths. If m is not zero and not n we could likewise delete a . Thus $m = 0$. If x has no neighbours on C_2 other than y , then consider instead of D_2 the cycle D_2' obtained from D_2 by replacing the portion of C_2 between c and y containing d by the portion of C_2 between c and y not containing d . Since $m = 0$, the net length of D_2' is n , so we can delete d and obtain a smaller weakly connected H' with two different non-zero net lengths. Otherwise, let y_1, y_2, \dots, y_p be all the neighbours of x on C_2 after $y = y_0$, numbered consecutively in the direction from y to c , away from d . Consider the oriented cycles Y_i containing x, y_i, y_{i+1} and the segment of C_2 between y_i and y_{i+1} not containing d . Each Y_i is an induced cycle in H , and the sum of their net lengths is n . Hence at least one Y_i has a non-zero net length and we similarly obtain a contradiction with the minimality of H .

Thus H consists of C_1, C_2 , and the two extra arcs (forward or backward) ad, cb ; in particular $u \in C_1 \cup C_2$, and the path U uses ad or bc . Without loss of generality, we may assume that it uses bc , since we can replace ad by ab, bc, cd . Suppose first that $u \in C_1$,

whence we also have $v \in C_1$. Consider the initial portion of W'_1 from v to $b = f(s_{i+1})$: it has net length equal to a multiple of k (corresponding to going full rounds around the cycle C_1) plus the net length of the portion X_1 of C_1 (in the positive direction) from u to b . Consider next the initial portion of W'_3 from v to c followed by the arc joining c and b : it has net length equal to n (corresponding to going from v to u , which must precede $c \in C_2$) plus a multiple of n (corresponding to going full rounds around the closed walk P_2 from u to u) plus the net length of the portion X_2 of P_2 (in the positive direction) from u to c concatenated with the arc joining c and b . However, from u to c we must use the arc joining b and c . Thus X_2 uses the arc joining b and c first in one direction and then in the opposite direction, whence the net lengths of X_1, X_2 are the same. This means that a multiple of n , smaller than $(a+2)n$ is also a multiple of k , which is impossible.

It remains to consider the case when W'_1, W'_3 do avoid each other. We now assume that of all homomorphisms f, g, h of S to Q_1, Q_2, Q_3 satisfying properties (1, 2, 3) and such that the resulting walks W'_1, W'_3 avoid each other, we have chosen ones that maximize the number of vertices with $f(s_i) = g(s_i)$ or $g(s_i) = h(s_i)$.

If W'_1, W'_3 have at least some faithful arcs, then Corollary 5.4 and Lemma 5.6 imply $\text{MinHOM}(H)$ is NP-complete. Thus we may assume that there are no faithful arcs between W'_1 and W'_3 .

We now define the costs of mapping vertices x of S to vertices j of H as follows: $c_j(x) = |S|$ except for $c_u(s) = c_u(t) = 1$, $c_v(s) = c_v(t) = 0$ and $c_j(s_i) = 0$ when $j \in \{f(s_i), g(s_i), h(s_i)\}$, $j \neq u$.

By properties (1, 2, 3), we see that to apply the Lemma 5.3 it remains to show that there is no homomorphism of S to H of cost $|S| - 1$ or less, taking both s and t to v . Suppose, for a contradiction, that there is such a homomorphism ϕ . Then we must have $\phi(s_0) = h(s_0)$, $\phi(s_q) = f(s_q)$, and each $\phi(s_i) \in \{f(s_i), g(s_i), h(s_i)\}$. Since there are no faithful arcs between W'_1 and W'_3 , we can't have $h(s_i)$ and $f(s_{i+1})$ adjacent. Thus, because of the costs, we must have some $h(s_i)$ and $g(s_{i+1})$ as well as $g(s_j)$ and $f(s_{j+1})$ are adjacent, with $i < j$. We now claim that this contradicts the maximality of f, g, h . Indeed, we could redefine f to equal g up to s_j (and then continuing as before, taking advantage of the arc joining $g(s_j)$ and $f(s_{j+1})$), obtaining a new W'_1 with at least one more vertex (namely s_{i+1}) having equality of f and g . (We need to observe that the new W'_1 still avoids W'_3 , which also follows by maximality of f, g, h : there cannot be an arc between $g(s_p) \neq h(s_p)$ and $h(s_{p+1})$.)

From the theorem we also derive the following corollary that will complete the proof of Theorem 4.2.

Theorem 5.11 *Suppose H is a digraph containing an induced oriented cycle of net length $k > 0$. If there is homomorphism $\ell : H \rightarrow \vec{C}_k$ with a sym- k -invertible pair, then $\text{MinHOM}(H)$ is NP-complete.*

Proof: Recall that P is a walk from u to v and Q a congruent walk with P , from v to u . Recall also that there is a homomorphism $\ell : H \rightarrow \vec{C}_k$, and $\ell(u) = \ell(v)$. It follows that the net length of P (and of Q) is divisible by k . If there are faithful arcs from P to Q or from Q to P then by Corollary 5.4 or 5.6, $\text{MinHOM}(H)$ is NP-complete. So we may assume that there are such faithful arcs. We may assume that the net length of P is greater than zero as otherwise remark following Lemma 5.8 implies that $\text{MinHOM}(H)$ is NP-complete. We now proceed to find congruent walks from u to v and from v to u which avoid each other, and another congruent walk from u to u , so that we can apply Lemma 5.3 in a fashion similar to what was done in the proof of Theorem 5.9.

We may assume that P is constricted from below, as otherwise we replace u, v by vertices $u' \in P, v' \in Q$, where u' is a vertex of P with the minimum height, and v' is the corresponding vertex of v' in Q . We have observed that u', v' is also a sym- k -invertible pair, thus there are walks P' from u' to v' and Q' from v' to Q' that avoid each other. It is easy to see that the minimality of u' implies that this new P' is constricted from below. Let C be a walk in H from u to an oriented cycle of net length k , followed by going around the oriented cycle once in the positive direction and then returning back on the same walk to u . Note that the net length of this walk is k . We may again assume that C is constricted from below, as otherwise instead of P, Q we could use P_1, Q_1 , where P_1 is obtained by concatenating P with $(QP)^a$ and Q_1 is obtained by concatenating Q with $(PQ)^a$ for some positive a , such that the walk from u (at the beginning of P_1) to the $(a - 1)$ -th appearance of u in P_1 , followed by C is a walk constricted from below.

Let the net length of P be ℓk , with $\ell > 0$. Let W be the infinite walk obtained by repeatedly concatenating C ; note that W is constricted from below. Let P' be the infinite walk obtained by concatenating P with infinitely many repetitions of QP . Let Q' be the infinite walk congruent to P' obtained by similarly concatenating Q with repetitions of PQ . Let C' be the walk in W , from u to a vertex u' that is the ℓ -th occurrence of u in W . Now we apply Corollary 5.10 to obtain a path $S = s_0, s_1, \dots, s_t$ which is the common pre-image of P, C', Q . In this application, we use P', W, Q' as the infinite walks, and the ends of P, C', Q as the vertices q_i . (Note that P, C', Q all have net length ℓk . Corollary 5.10 also yields homomorphisms f, g, h of S to P', W, Q' taking s_0 to the beginnings of P', W, Q' (also the beginnings of P, C', Q), and taking s_t to the ends of P, C', Q . Let P'' be the walk $f(s_0), f(s_1), \dots, f(s_t)$, let Q'' be the walk $h(s_0), h(s_1), \dots, h(s_t)$, and let C'' be the walk $g(s_0), g(s_1), \dots, g(s_t)$. Observe that P'', Q'' avoid each other and between the walks P'', Q'' there are no faithful arc, because that was the case for P, Q .

Note that $f(s_0) = u$ and $f(s_t) = v$, $g(s_0) = g(s_t) = u$ and $h(s_0) = v, h(s_t) = u$. We define the costs as follows, the $c_u(s_0) = c_u(s_t) = 1$, and $c_v(s_0) = c_v(s_t) = 0$, and $c_i(x) = 0$ when $i \in \{f(x), g(x), h(y)\}$, $x \neq u$. For any other case the cost is $|V(S)|$.

We now conclude the proof as in Theorem 5.9, assuming that the homomorphisms f, g, h of S to $V(P'') \cup V(C'') \cup V(Q'')$ satisfy properties 1, 2, 3, and maximize the number of vertices with $f(s_i) = g(s_i)$ or $g(s_i) = h(s_i)$. \diamond

We are finally ready to conclude the **Proof of Theorem 4.2**, i.e., to prove the conjecture from [18].

Recall that the polynomial case of the Theorem has been established in [18]. For the NP-completeness claim, the case when H is balanced is handled by Corollary 5.8. Thus we may assume that H has an induced oriented cycle of some positive net length k . It is a well-known fact (e.g. Corollary 1.17 in [24]) that H has a homomorphism to \vec{C}_k if and only if it does not contain a closed walk of net length not divisible by k . Suppose first that H does not admit a homomorphism to \vec{C}_k . Then the above fact implies that H contains an induced oriented cycle of net length not divisible by k . Hence the problem $\text{MinHOM}(H)$ is NP-complete by Theorem 5.9. If, on the other hand, H does admit a homomorphism to \vec{C}_k , with a sym- k -invertible pair, then $\text{MinHOM}(H)$ is NP-complete by Theorem 5.11. This completes the proof.