Lecture 4. Postulates of quantum mechanics

The entire theory of quantum mechanics can be derived from the following 4 postulates.

Postulate 1 (State space)
Associated to any isolated physical system is a Hilbert space, called the state space. The system is completely described by its state vector, which is a unit vector in the system’s state space.

Postulate 2 (Evolution)
The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\psi\rangle$ of the system at time $t_1$ is related to the state $|\psi'\rangle$ of the system at time $t_2$ by a unitary operator $U$, which depends only on the times $t_1$ and $t_2$

$$|\psi'\rangle = U|\psi\rangle.$$ 

This postulate can be transformed to adopt continuous time

Postulate 2' (Schrödinger Equation)
The time evolution of the state of closed quantum system is described by the Schrödinger equation

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle,$$

where $\hbar$ is Planck’s constant, and $H$ is the Hamiltonian of the system. The Hamiltonian is always a Hermitian operator.

Since $H$ is Hermitian, it has a spectral decomposition

$$H = \sum_E |E\rangle\langle E|,$$

with eigenvalues $E$ and corresponding normalized eigenvectors $|E\rangle$. The eigenvalues correspond to energy eigenstates or stationary states, and $E$ is the energy of the state $|E\rangle$.

Solving the Schrödinger equation we get

$$|\psi(t_2)\rangle = \exp\left(-\frac{iH(t_2 - t_1)}{\hbar}\right)|\psi(t_1)\rangle = U(t_1, t_2)|\psi(t_1)\rangle,$$

and the corresponding unitary transformation is

$$U(t_1, t_2) = \exp\left(-\frac{iH(t_2 - t_1)}{\hbar}\right).$$
Postulate 3 (Composite Systems)
The state space of a composite physical system is the tensor product of the component systems. Moreover, if we have systems numbered 1 through $n$, and the system number $i$ is prepared in state $|\psi_i\rangle$, then the joint state of the combined system is $|\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$.

Postulate 4 (Measurement)
Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operators acting on the state space of the system being measured. The index $m$ refers to the measurement outcomes that may occur in the experiment. If the state of the system is $|\psi\rangle$ immediately before the measurement, then the probability that result $m$ occurs is given by

$$P(m) = \langle \psi | M_m^\dagger M_m |\psi\rangle,$$

and the state of the system after the measurement is

$$\frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m |\psi\rangle}}.$$

The measurement operators satisfy the completeness condition

$$\sum_m M_m^\dagger M_m = 1.$$

Projective measurement.
Projective measurements is an important particular class of measurements. Let measurement operators $\{M_m\}$ be defined by $M_m = mP_m$, where $P_m$ are projectors onto pairwise orthogonal subspaces.
Operator $M = \sum_m mP_m$ is called an observable. Then the $m$’s are eigenvalues of $M$, the subspaces are eigenspaces of $M$, and from the completeness condition they span the entire space.

The probability of outcome $m$ equals

$$P(m) = \langle \psi | P_m |\psi\rangle.$$

Given that outcome $m$ occurred, the state of the system immediately after the measurement is

$$\frac{P_m |\psi\rangle}{\sqrt{P(m)}}.$$

Distinguishability of quantum states.
Theorem 1 (Indistinguishability theorem) Two states can be reliably distinguished if and only if they are orthogonal.

Proof: If \(|\psi_1\rangle, |\psi_2\rangle\) are orthogonal, then choose an orthonormal basis \(|a_1\rangle, \ldots, |a_n\rangle\) such that \(|a_1\rangle = |\psi_1\rangle\) and \(|a_2\rangle = |\psi_2\rangle\). The measure using observable

\[ M = \sum_{i=1}^{n} iP_{|a_i\rangle}. \]

Suppose that \(|\psi_1\rangle\) and \(|\psi_2\rangle\) are non-orthogonal, and suppose that \(\{M_m\}\) allow one to distinguish \(|\psi_1\rangle\) and \(|\psi_2\rangle\). This means that there is a function \(f: \{m_1, \ldots, m_k\} \to \{0, 1, 2\}\) such that if we have \(|\psi_i\rangle\) and outcome is \(m_j\) then \(f(m_j) = i\). We set

\[ E_i = \sum_{j: f(j) = i} M_j^\dagger M_j. \]

Then since in \(|\psi_i\rangle\) one of the \(j: f(j) = i\) happens with probability 1, we have

\[ \langle \psi_i | E_i | \psi_i \rangle = 1. \]

Hence \(|\psi_1\rangle E_2 |\psi_1\rangle = 0\).

Let \(|\psi_2\rangle = \alpha |\psi_1\rangle + \beta |\varphi\rangle\) where \(|\varphi\rangle\) is orthogonal to \(|\psi_1\rangle\) and \(\alpha \neq 0\). Then

\[ \langle \psi_2 | E_2 | \psi_2 \rangle = |\alpha|^2 \langle \psi_1 | E_2 | \psi_1 \rangle + |\beta|^2 \langle \varphi | E_2 | \varphi \rangle \]

\[ = |\beta|^2 \langle \varphi | E_2 | \varphi \rangle \leq |\beta|^2 < 1. \]

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