Complexity of Non-Uniform CSP

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Constraint Satisfaction Problem

Definition: \( \text{CSP}(A) \)

Instance: \((V; A; C)\) where

- \(V\) is a finite set of variables
- \(A\) is a finite set of similar finite algebras
- \(C\) is a set of constraints \(\{R_1(s_1), \ldots, R_q(s_q)\}\) where each \(R_i\) is a subalgebra of a direct product of algebras from \(A\)

Question: whether there is \(h: V \rightarrow \bigcup A\) such that, for any \(i\), \(R_i(h(s_i))\) is true
Constraint Satisfaction Problem

Definition:
Instance: \((V; \mathcal{A}; \mathcal{C})\) where
- \(V\) is a finite set of variables
- \(\mathcal{A}\) is a set of finite domains
- \(\mathcal{C}\) is a set of constraints \(\{R_1(s_1), \ldots, R_q(s_q)\}\) where each \(R_i\) is a relation over a Cartesian product of sets from \(\mathcal{A}\)

Question: whether there is \(h: V \rightarrow \bigcup \mathcal{A}\) such that, for any \(i\), \(R_i(h(s_i))\) is true
CSP and Friends

- CSP
- Non-Uniform CSP
- homomorphism problems
Homomorphism Problems

**Homomorphism Problem:**
Given relational structures $G$ and $H$ of the same type, decide, whether or not $G \rightarrow H$

Equivalent to CSP:
- $G$: elements are variables, tuples are constraint scopes
- $H$: elements are elements, relations are (constraint) relations

**H-Coloring**: ($H$ is a fixed structure)
Given $G$, decide whether $G \rightarrow H$
**Example: Graph Homomorphism, H-Coloring**

**k-Coloring:**
- **Instance:** A graph $G$.
- **Objective:** Is there a $k$-coloring of $G$?

Is there a homomorphism from $G$ to $K_k$?

![Graphs](image-url)
Homomorphism Problems II

Instead of fixing \( H \), restrict possible \( G \)

Example: Problems on planar graphs

Vardi:
- Query complexity: fix \( H \)
- Data complexity: restrict \( G \)
CSP and Friends

databases
Datalog

CSP
Non-Uniform CSP

homomorphism
problems
Databases

(Relational) Database:
A bunch of relations

Query:
A logic formula $\Phi$. Enumerate all models of $\Phi$ in the database

Conjunctive query:
\[ R_1(x, y) \land R_2(z, x, x) \land \ldots \]

Conjunctive queries $= \text{(enumeration) CSP}$
Databases: Query Containment and Equivalence

Conjunctive query is a homomorphism problem
\[ \Phi \rightarrow B \]

How about C.Q. \( \Phi_1, \Phi_2 \)?

We say \( \Phi_1 \) is contained in \( \Phi_2 \) (\( \Phi_1 \leq \Phi_2 \)) if every answer to \( \Phi_1 \) is an answer to \( \Phi_2 \)

Queries \( \Phi_1, \Phi_2 \) are equivalent if \( \Phi_1 \leq \Phi_2 \) and \( \Phi_2 \leq \Phi_1 \)

Chandra-Merlin:
\[ \Phi_1 \leq \Phi_2 \quad \text{iff} \quad \Phi_1 \rightarrow \Phi_2 \]
**Datalog**

*Datalog* is `logic language` simulating the `least fixed point` operator

\begin{align*}
P(x,y) & : - E(x,y) \\
P(x,y) & : - P(x,z), E(z,t), E(t,y) \\
R(x) & : - P(x,x)
\end{align*}

*Datalog* gives CSPs solvable by local propagation algorithms

Barto-Kozik: For non-uniform CSPs being solvable by *Datalog* is equivalent to a nice algebraic condition
CSP and Friends

databases
  Datalog

the other side

CSP
  Non-Uniform CSP

homomorphism problems
The Other Side

Let $\mathcal{G}$ be a class of structures

$\text{CSP}(\mathcal{G},*)$:
Given $\mathcal{G} \in \mathcal{G}$ and any $\mathcal{H}$, decide whether $\mathcal{G} \rightarrow \mathcal{H}$

Grohe: For a class $\mathcal{G}$ of structures of bounded arity

$\text{CSP}(\mathcal{G},*)$ is poly time iff the cores of structures from $\mathcal{G}$ have bounded treewidth (mod some complexity assumptions)

This condition can also be expressed through some logic games, homomorphism duality, etc.
The Other Side II

Marx: For a class $G$ of structures $\text{CSP}(G,*)$
- is poly time if $G$ has bounded fractional hypertree width
- is `fixed parameter tractable’ if $G$ has bounded submodular width
- `very hard’ otherwise (mod some complexity assumptions)
CSP and Friends

CSP
Non-Uniform CSP

- databases
  - Datalog
- the other side
- homomorphism problems
- Logic:
  - dichotomies
  - logic for P
CSP vs. NP

Fagin: NP is the class of problems expressible in the existential second order logic (ESO)
If P ≠ NP there are infinitely many intermediate complexity classes (no dichotomy)
How much do we need to restrict NP to have a dichotomy?

Valiant, Cai: for counting problems
Marx: combinatorial conditions
Feder/Vardi: MMSNP

Feder/Vardi, Kun:
MMSNP is poly time equivalent to CSP
Logic for P

No Fagin’s theorem for P

$\text{FO}$ is very weak

$LFP(\text{FO})$ (think Datalog)
  Gurevich: expresses all in $\text{P}$ provided structures are ordered
  otherwise does not work for linear algebra

$LFP(\text{FO})$+counting quantifiers
  Still does not express matrix rank

$LFP(\text{FO})$+counting+rank operator
CSP and Friends

- valued CSPs
- databases
- Datalog
- the other side
- homomorphism problems
- Logic:
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  - logic for P

Non-Uniform CSP
Valued CSPs

MaxCSP/MinCSP:

Given a CSP instance, satisfy as many constraints as possible / unsatisfy as few as possible

Valued CSPs:

Same as MinCSP, except every tuple in a constraint has a (numerical) value, and we need to minimize the total value of such tuples produced by an assignment
Valued CSP: Complexity

Zivny/Thapper:
Without crisp constraints, the only poly time algorithm is linear programming

Kolmogorov/Krokhin/Rolinek:
With crisp constraints, LP+whatever algorithm for CSP is the best that can be done
CSP and Friends

approximation
UGC

databases
Datalog

the other side

CSP
Non-Uniform CSP

valued CSPs

homomorphism problems

Logic:
- dichotomies
- logic for P
Approximation

Approximation algorithms and complexity is a big area

Often we are talking about approximating a MaxCSP or a Valued CSP
Approximation: Unique Games Conjecture

Consider a CSP with binary constraints

\[ R_1(x, y) \land R_2(z, x) \land \ldots \]

where each relation is the graph of a permutation

**Unique Games Conjecture (Khot):**

Such a CSP is absolutely impossible to approximate

**Raghavendra:**

Assuming UGC, an optimal approximation algorithm for any CSP without crisp constraint
CSP and Friends

- approximation
  - UGC
- valued CSPs
- databases
  - Datalog
- the other side
- homomorphism problems
- A lot of other stuff:
  - social choice
  - long codes
  - decay of correlations
  - see abstracts of this conf
- Logic:
  - dichotomies
  - logic for P
Now the talk begins
Dichotomy conjecture and theorem

**Theorem**
For any finite class $A$ of finite similar algebras the problem $\text{CSP}(A)$ is either solvable in polynomial time or NP-complete.

It suffices to prove the theorem for idempotent algebras

**Theorem**
For any finite class $A$ of finite similar idempotent algebras the problem $\text{CSP}(A)$ is solvable in polynomial time if $A$ has a WNU. It is NP-complete otherwise.
Two Main Algorithms

- Local propagation algorithms: Datalog (Vardi, Kolaitis, Dalmau, Barto, Kozik, B., ...)

- Few subalgebras: edge term, generating set for solutions (B., Dalmau, Berman, Idziak, Markovič, McKenzie, Valeriote, Kearns, Szendrei)
Ingredients

- Separation of prime congruence intervals
- Semilattice edges
- Algorithm
Separation of prime congruence intervals

Let $R$ be a subdirect product of $A_1 \times \cdots \times A_n$,
let $i, j \in \{1, \ldots, n\}$ and $\alpha < \beta$, $\gamma < \delta$ prime intervals in $Con(A_i)$ and $Con(A_j)$, respectively.

We say that $\alpha < \beta$ can be separated from $\gamma < \delta$, if there is a polynomial $f$ of $R$ such that $f(\beta) \not\subseteq \alpha$ while $f(\delta) \subseteq \gamma$. 
Coherent Sets

Let $P = (V, A, C)$ be an instance.

Let $v \in V$ and $\alpha < \beta$ a prime interval in $\text{Con}(A_v)$.

The set $W = W(v, \alpha, \beta)$ of all $w \in V$ such that $\text{Con}(A_w)$ contains $\gamma < \delta$ such that $\alpha < \beta, \gamma < \delta$ cannot be separated is called a coherent set.
Coherent Sets II

Let $P = (V, A, C)$ be an instance.

$P_W$ is a restricted problem $(W, A, C|_W)$:

$R(s) \rightarrow pr_{s \cap W}R(s \cap W)$

**Condition (QC):** some commutator-like condition of a prime interval in a congruence lattice

**Theorem**

If $\alpha < \beta$ does not satisfy Condition (QC) then $P_W$ can be decomposed into a constant number of instances over smaller domains
Splitting Instances

Let $P_w$ be as before and $\alpha_w < \beta_w$ prime interval in $Con(A_w)$ such that $\alpha_w < \beta_w$ cannot be separated from $\alpha_u < \beta_u$ for any $u, w \in W$

There are $\theta_w \in Con(A_w)$ such that $P_w$ is $\bar{\theta}$-linked, that is, for any $u, w \in W$ and $\bar{a}, \bar{b} \in P_{u,w}$ if $(a_u, b_u) \in \theta_u$ then $(a_w, b_w) \in \theta_w$
Ingredients

- Separation of prime congruence intervals
- Semilattice edges
- Algorithm
Semilattice Edges

Let $A$ be an algebra. A pair $a, b \in A$ is said to be a semilattice edge if there is a term operation $\cdot$ of $A$ which is semilattice on $\{a, b\}$, i.e.

- $a \cdot a = a$
- $a \cdot b = b \cdot a = b \cdot b = b$

Operation $\cdot$ can be chosen such that it is semilattice on all semilattice edges of all algebras from $A$

Algebra $A$ is semilattice free if it does not have a semilattice edge
Ingredients

- Separation of prime congruence intervals
- Semilattice edges
- Algorithm
Algorithm: Assumptions

Let $P = (V,\mathcal{A},\mathcal{C})$ be an instance.
We will assume:

- every non-semilattice free domain of $P$ is subdirectly irreducible,
  let $\mu_{\nu}$ denote the monolith of $A_{\nu}$
Algorithm: Max and Center

Let $P = (V, A, C)$ be an instance

$max(P)$ is the maximal size of domains of $P$ with a semilattice edge

$Max(P) \subseteq V$ is the set of variables whose domains are not semilattice free and have size $max(P)$

$Center(P) \subseteq V$ is the set of variables $v \in V$ such that $0_v < \mu_v$ satisfies Condition (QC)
Algorithm: Cases

Let $P = (V, A, C)$ be an instance

Recursion on $\max(P)$

We consider 3 cases

(A) All the domains in $P$ are semilattice free

(B) $\text{Max}(P) \cap \text{Center}(P) = \emptyset$

(C) $\text{Max}(P) \cap \text{Center}(P) \neq \emptyset$
Algorithm: Case (A)

Theorem
Let $A$ be a semilattice free algebra. Then $A$ has few subpowers

Suppose all the domains in $P$ are semilattice free
Then $P$ can be solved by the few subpowers algorithm
Quotient Problem

Let \( P = (V, A, C) \) be an instance

\( P_{W/\bar{\mu}} \) is the problem \( (V, A/\bar{\mu}, C/\bar{\mu}) \), where

\[
R(s) \rightarrow R/\bar{\mu}(s)
\]
Algorithm: Block-Minimality

Let $P = (V, A, C)$ be an instance. It is called block-minimal, if for every $v \in V$ and every $\alpha < \beta \in Con(A_v)$

- if $\alpha < \beta$ does not satisfy Condition (QC), $P_W$, $W = W(v, \alpha, \beta)$, is minimal
- if $\alpha < \beta$ satisfies Condition (QC), then $P_W/\bar{\mu}$ is minimal

Observation: Establishing block minimality is done by solving polynomially many smaller instances
Algorithm: Case (B) - Empty Center

Theorem
Let $P = (V, A, C)$ be a block-minimal instance. If $\text{Max}(P) \cap \text{Center}(P) = \emptyset$ then $P$ has a solution.
Algorithm: Case (C) - Nonempty Center

Let $\alpha^*_v$ be $\mu_v$ if $v \in Max(P) \cap Center(P)$, and $0_v$ otherwise

**Theorem**

Let $P = (V, A, C)$ be a block-minimal instance.

1. There is a solution $\varphi$ of $P' = P/\alpha^*$ such that for every $v \in V$ for which $A_v$ is not semilattice free, there is a $\alpha^*_v$-block $B_v$ such that $B_v, \varphi(v)$ is a semilattice edge.

2. Instance $P'' = P \cdot \varphi$ is equivalent to $P$ and such that $\max(P'') < \max(P)$. 
Thank you!
Ingredients

- Separation of prime congruence intervals
- Quasi-Centralizers
- Semilattice edges
- Strategies
Separation of prime congruence intervals

Let $A$ be an algebra and $\alpha < \beta$, $\gamma < \delta$ prime intervals in $Con(A)$

We say that $\alpha < \beta$ can be separated from $\gamma < \delta$, if there is a polynomial $f$ of $A$ such that $f(\beta) \not\subseteq \alpha$ while $f(\delta) \subseteq \gamma$

Let $R$ be a subdirect product of $A_1 \times \cdots \times A_n$, let $i, j \in \{1, \ldots, n\}$ and $\alpha < \beta$, $\gamma < \delta$ prime intervals in $Con(A_i)$ and $Con(A_j)$, respectively

We say that $\alpha < \beta$ can be separated from $\gamma < \delta$, if there is a polynomial $f$ of $R$ such that $f(\beta) \not\subseteq \alpha$ while $f(\delta) \subseteq \gamma$
Collapsing polynomials

Let $R$ be a subdirect product of $A_1 \times \cdots \times A_n$, let $\alpha < \beta$ be a prime interval in $\text{Con}(A_1)$ be such that $\alpha < \beta$ can be separated from EVERY interval $\gamma < \delta$ from $\text{Con}(A_j)$ for EVERY $j \neq 1$.

Then there is a polynomial $f$ of $R$ such that
- $f(\beta) \not\subseteq \alpha$
- $|f(A_j)| = 1$ for every $j \neq 1$
Quasi-Centralizers

Let $A$ be an algebra and $\alpha < \beta$ prime intervals in $Con(A)$, $\chi(\alpha, \beta)$ denotes the binary relation on $A$ given by:

$$(a, b) \in \chi(\alpha, \beta) \text{ iff for any term } f(x, y, z_1, ..., z_n) \text{ and any } c_1, ..., c_n \in A: \ g(\beta) \subseteq \alpha \iff h(\beta) \subseteq \alpha,$$

where $g(x) = f(x, a, c_1, ..., c_n)$ and $h(x) = f(x, b, c_1, ..., c_n)$

It is a congruence of $A$
Splitting Relations

Let $R$ be a subdirect product of $A_1 \times A_2$ and $\alpha < \beta$, $\gamma < \delta$ prime intervals in $\text{Con}(A_1)$, $\text{Con}(A_2)$, respectively, such that they cannot be separated from each other. Also, let $\theta_1 = \chi(\alpha, \beta), \theta_2 = \chi(\gamma, \delta)$. Then $R$ is $\bar{\theta}$-linked, that is, for any $(a, b), (c, d) \in R$ if $(a, c) \in \theta_1$ then $(b, d) \in \theta_2$ and the other way round.
Splitting Relations II

Let $R$ be a subdirect product of $A_1 \times \cdots \times A_n$ and $\alpha_i < \beta_i$ prime interval in $Con(A_i)$ such that $\alpha_i < \beta_i$ cannot be separated from $\alpha_j < \beta_j$ for any $i, j$.

Also, let $\theta_i = \chi(\alpha_i, \beta_i)$,

Then $R$ is $\bar{\theta}$-linked, that is, for any $\bar{a}, \bar{b} \in R$ if

$(a_i, b_i) \in \theta_i$ then

$(a_j, b_j) \in \theta_j$ for any $i, j$
Coherent Sets

Let $P = (V, A, C)$ be a $(2,3)$-minimal instance.
Let $v \in V$ and $\alpha < \beta$ a prime interval in $\text{Con}(A_v)$
The set $W = W(v, \alpha, \beta)$ of all $w \in V$ such that $\text{Con}(A_w)$ contains a prime interval $\gamma < \delta$ and $\alpha < \beta, \gamma < \delta$ cannot be separated from each other.

Theorem
If $\chi(\alpha, \beta)$ is not the full congruence, $P_W$ can be decomposed into a constant number of instances over smaller domains.
Semilattice Edges

Let \( A \) be an algebra. A pair \( a, b \in A \) is said to be a semilattice edge if there is a term operation \( \cdot \) of \( A \) which is semilattice on \( \{a, b\} \), i.e.

- \( a \cdot a = a \)
- \( a \cdot b = b \cdot a = b \cdot b = b \)

Operation \( \cdot \) can be chosen such that it is semilattice on all semilattice edges of all algebras from \( A \).

For any \( a, b \in A \) either \( a \cdot b = a \) or \( a, a \cdot b \) is a semilattice pair.
Semilattice Edges II

**Theorem**

Let $A$ be an algebra and $\alpha < \beta \in Con(A)$ such that $\beta \leq \chi(\alpha, \beta)$. For any $a, b, c \in A$ such that $(b, c) \in \beta$ and $(a, b) \in \chi(\alpha, \beta)$, it holds $(a \cdot b, a \cdot c) \in \alpha$.

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**Diagram:**

- **Semilattice edges in $A/\alpha$**
- **$\beta$-blocks $B, C$ such that $B, C$ is a semi. edge in $A/\beta$**
Algorithm: Standard Reductions

Let $P = (V, A, C)$ be an instance
We will assume:

- $P$ is (2,3)-minimal
- every its domain is subdirectly irreducible
  let $\mu_v$ denote the monolith of $A_v$
Algorithm: Max and Center

Let $P = (V, A, C)$ be an instance

$\text{max}(P)$ is the maximal size of semilattice free domains of $P$

$\text{Max}(P) \subseteq V$ is the set of variables whose domains are semilattice free and have size $\text{max}(P)$

$\text{Center}(P) \subseteq V$ is the set of variables $\nu \in V$ such that $\chi(0_{\nu}, \mu_{\nu})$ is the full congruence
Algorithm: Cases

Let \( P = (V, A, C) \) be an instance

We consider 3 cases

(A) All the domains in \( P \) are semilattice free

(B) \( \text{Max}(P) \cap \text{Center}(P) = \emptyset \)

(C) \( \text{Max}(P) \cap \text{Center}(P) \neq \emptyset \)
Algorithm: Case (A)

**Theorem**

Let $A$ be a semilattice free algebra. Then $A$ has few subpowers

Suppose all the domains in $P$ are semilattice free
Then $P$ can be solved by the few subpowers algorithm
Algorithm: Block-Minimality

Let $P = (V, A, C)$ be an instance
It is called block-minimal, if

- for every $v \in V$ and every $\alpha < \beta \in \text{Con}(A_v)$

- if $\chi(\alpha, \beta)$ is not the full congruence, $P_W$, $W = W(v, \alpha, \beta)$, is minimal

- if $\chi(\alpha, \beta)$ is the full congruence, then $P_W / \bar{\mu}$ is minimal

Observation: Establishing block minimality is done by solving polynomially many smaller instances
Algorithm: Case (B) - Empty Center

**Theorem**
Let $P = (V, A, C)$ be a block-minimal instance. If $\text{Max}(P) \cap \text{Center}(P) = \emptyset$ then $P$ has a solution.
Algorithm: Case (C) - Nonempty Center

Let $\alpha^*_v$ be $\mu_v$ if $v \in \text{Max}(P) \cap \text{Center}(P)$, and $0_v$ otherwise

Theorem

Let $P = (V,A,C)$ be a block-minimal instance.

1. If $P = P/\alpha^*$ is 1-minimal then there is a solution $\varphi$ of $P'$ such that for every $v \in V$ such that $A_v$ is not semilattice free there is a $\alpha^*_v$-block $B_v$ such that $B_v, \varphi(v)$ is a semilattice edge.

2. Instance $P'' = P \cdot \varphi$ is equivalent to $P$ and such that $\text{max}(P'') < \text{max}(P)$
Theorem
Let $P = (V, A, C)$ be a block-minimal instance. If $\text{Max}(P) \cap \text{Center}(P) = \emptyset$ then $P$ has a solution.

We show that for any $\beta_v \in \text{Con}(A_v)$ there is a solution of $P/\beta$.
If $\beta_v$ is the full congruence, such a solution exists.
If $\beta_v = 0_v$ then we have a solution of $P$. 

Strategies I
Strategies II

Let $\beta_v \in \text{Con}(A_v)$ and $B_v$ a $\beta_v$-block
$W(\bar{\beta})$ is the set of triples $(v, \alpha, \beta)$, where $v \in V$, $\alpha < \beta \leq \beta_v \in \text{Con}(A_v)$
Let $R$ be a collection of relations $R_{C,v,\alpha\beta}$ for each constraint $C = \langle s, R \rangle \in \mathcal{C}$ and $(\alpha, \beta) \in W(\bar{\beta})$
Let $S(C, v, \alpha\beta) = s \cap W(v, \alpha, \beta)$ be the set of its coordinate positions
A tuple $\alpha \in \prod_{x \in X} A_x$ for $X \subseteq V$ is said to be $R$-compatible if for any $C = \langle s, R \rangle \in \mathcal{C}$ and $(\alpha, \beta) \in W(\bar{\beta})$
$pr_T \alpha \in pr_T R_{C,v,\alpha\beta}$, where $T = X \cap S(C, v, \alpha\beta)$
Strategies III

\( R \) is said to be a \( \bar{\beta} \)-strategy with respect to \( \bar{B} \) if for every \( C = \langle s, R \rangle \in C \) and \( (v, \alpha, \beta) \in W(\bar{\beta}) \) the following conditions hold (\( W = W(v, \alpha \beta) \)):

(S1) the relations \( R^X, R, X \subseteq V, |X| \leq 2 \), consisting of \( R \)-compatible tuples from \( R^X \), form a nonempty \((2,3)\)-strategy for \( P \)

(S2) for every \( (w, \gamma, \delta) \in W(\bar{\beta}) \) (let \( U = W(v, \alpha, \beta) \)) and every \( a \in pr_{s \cap W \cap U} R_{C,v,\alpha \beta} \) tuple \( a \) extends to an \( R \)-compatible solution of \( P_U \)
Strategies IV

(S3) \( R \cap \prod_w B_w \neq \emptyset \) and for any \( I \subseteq s \) any \( \mathcal{R} \)-compatible tuple \( \mathbf{a} \in \text{pr}_I R \) extends to an \( \mathcal{R} \)-compatible tuple from \( \mathcal{R} \)
Tightening Strategies

Theorem

Let $R$ be a $\bar{\beta}$-strategy with respect to $\bar{B}$.
Let $(\nu, \alpha, \beta) \in W(\bar{\beta})$ be such that $\alpha|_{B_\nu} \neq \beta|_{B_\nu}$ and $\beta = \beta_\nu$. Set $\beta'_\nu = \alpha$ and $\beta'_w = \beta_w$.
Let $B'_\nu \subseteq B_\nu$ be an $\alpha$-block.
Then there is a $\bar{\beta}'$-strategy with respect to $\bar{B}'$. 