Circular Chromatic Number of Hypergraphs

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Abstract

The concept of circular chromatic number of graphs was introduced by Vince(1988). In this paper we define circular chromatic number of uniform hypergraphs and study their basic properties. We study the relationship between circular chromatic number with chromatic number and fractional chromatic number of uniform hypergraphs.

Keywords : Hypergraph, Circular Coloring

1 Introduction

Vince introduced the following generalization of chromatic number [9]. For necessary definitions and notations we refer the reader to standard texts such as [10]. In this paper p, q, r, k, m, n, l are positive integers and $k > 2$.

Definition 1.1 [9]. Let $G = (V, E)$ be a graph. A (p, q)-*coloring*, $(2q \leq p)$, of G is a mapping $c : V \longrightarrow \{0, 1, .., p-1\}$ such that, if $xy \in E(G)$ then $q \leq |c(x) - c(y)| \leq p - q$. The *circular chromatic number* of G, $\chi_c(G)$, is defined as $\chi_c(G) = inf\{\frac{p}{q} \mid \ there \ \ is \ \ a \ \ (p,q) - coloring \ \ for \ \ G\}$.
Vince [9] showed that for finite graph G, the infimum in the definition is attained, thus $\chi_c(G)$ is rational. He also proved that for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$.

Bondy and Hell [3] proved that if $\frac{p}{q} \leq \frac{p'}{q'}$, and $\chi_c(G) = \frac{p}{q}$ then G has a (p', q')-coloring. And if $(p, q) = 1$ then in every (p, q)-coloring of G every color appears, and $p \leq |V(G)|$.

Let G^p_q denote the graph with vertex set $\{0, 1, .., p-1\}$ and i is adjacent to j if $q \leq |i - j| \leq p - q$. Vince [9] proved that $\chi_c(G^p_q) = \frac{p}{q}$ and Zhu [11] showed that for any vertex v of G^p_q, $G^p_q - v$ has circular chromatic number strictly less than $\frac{p}{q}$. Bondy and Hell [3] showed that

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$\chi_c(G) = min\{\frac{p}{q}|$ *there exists a homomorphism* $f : G \longrightarrow G^p_q\}$. By this equivalent definition we conclude that if $f : G \longrightarrow G'$ is a homomorphism then $\chi_c(G) \leq \chi_c(G')$.

The question that for which graphs G, $\chi_c(G) = \chi(G)$ was raised in [9]. It was shown by Guichard [5] that it is NP-hard to determine whether or not an arbitrary graph G satisfies $\chi_c(G) = \chi(G)$. There are some sufficient conditions for which circular chromatic number and chromatic number of a graph are equal. For more information one of the good references is [12]. Beside the chromatic number of a graph other graph parameters whose relationship with circular chromatic number have been investigated. One of them is the fractional chromatic number. It was shown that $\chi_f(G) \leq \chi_c(G)$.[12]

In section 2 we introduced some definitions that we need. In section 3 we define circular chromatic number of uniform hypergraphs. In section 4 we study relation between chromatic number and circular chromatic number of uniform hypergraphs.

2 Preliminaries and Definitions

In this section some definition are introduced and used in the other sections. In this paper we consider finite hypergraphs.

Definition 1 *Let $H = (V, E)$ be a hypergraph with vertex set V and edge set E. H is called k-uniform if for every edge e of E, we have $|e| = k$.*

Definition 2 *A set $S \subset V(H)$ is called independent if there is no edge e of H contained in S. The independence number of H, $\alpha(H)$, is the size of a greatest independent set in H.*

Definition 3 *A k-uniform hypergraph H is called complete if every k-subset of $V(H)$ be contained in $E(H)$.*

In the following we define a homomorphism between two hypergraphs H and K :

Definition 4 *A mapping $f : V(H) \longrightarrow V(K)$ is called homomorphism if for every edge e of H, there exists an edge e' of K such that $e' \subseteq f(e)$. We say H is homomorphic to K, if there is a homomorphism of H to K.*

Definition 5 *A hypergraph H is called vertex transitive, if for every vertices x and y in H there is an bijection homomorphism $f : H \longrightarrow H$ such that $f(x) = y$.*

Definition 6 *Let $H_q^p(k)$ denote the k-uniform hypergraph with vertex set $\{0, 1, .., p-1\}$ and a k-subset $\{x_1, x_2, .., x_k\}$ of $V(H)$ is an edge of $H_q^p(k)$ if and only if there exist $1 \leq i, j \leq k$ such that $q \leq |x_i - x_j| \leq p - q$.*

Now we generalize the concept of fractional coloring and fractional chromatic number of hypergraph.

Definition 7 *A mapping c from the collection S of independent sets of a hypergraph H to the interval $[0, 1]$ is a fractional-coloring of H if for every vertex x of H we have $\Sigma_{s \in S, x \in s} c(s) = 1$. The value of fractional-coloring c is $\Sigma_{s \in S} c(s)$. The fractional coloring number $\chi_f(H)$ of H is the infimum of the values of fractional-colorings of H.*
Similar to graphs the infimum in the definition is attend and we can can replace the infimium by minimum.

Definition 8 *A vertex x is called universal if every k- subset of $V(H)$ containing x, is an edge of H.*

Definition 9 *A hypergraph H is called uniquely n-colorable if $\chi(H) = n$ and any n-coloring of H produce the same partition for $V(H)$.*

3 Circular Coloring of Hypergraph

In this section we generalize (p, q)-coloring and circular chromatic number of hypergraph.

Definition 10 *Let $H = (V, E)$ be a hypergraph. A (p, q)-coloring, $(2q \leq p)$, of H is a mapping $c : V \longrightarrow \{0, 1, .., p - 1\}$ such that, for every edge $e \in E$, there exist x and y in e satisfying $q \leq |c(x) - c(y)| \leq p - q$.*
The circular chromatic number of H, $\chi_c(H)$, is defined as

$$\chi_c(H) = inf\{\frac{p}{q} \mid \; there \; exists \; a \; (p, q) - coloring \; of \; H\}$$

By a similar proof for finite graphs we can prove that the "infimum" can be replaced by "minimum".

Proposition 1 *Let H be a hypergraph then $\chi_c(H) = min\{\frac{p}{q}|$ there exists a homomorphism $f : H \longrightarrow G_q^p\}$.*

Proof: Let c be a (p, q)-coloring of H then the mapping $f : V(H) \longrightarrow V(G_q^p)$ defined by $f(x) = c(c)$ is a homomorphism and converse holds. □

The following theorem is a consequence of the above definition, and Proposition 1.

Theorem 2 *Let H and K be two hypergraphs and H is homomorphic to K then,*

a) $\chi(H) \leq \chi(K)$,

b) $\chi_c(H) \leq \chi_c(K)$.

Theorem 3 *Let H be a k-uniform hypergraph and $\chi_c(H) = \frac{p}{q}$ then*

a) $\chi(H) - 1 < \chi_c(H) \leq \chi(H)$,

b) Let $c : V(H) \longrightarrow \{0, 1, .., p-1\}$ be a (p, q)-coloring of H. Then c is onto, and $|V(H)| \geq p$.

c) If $\frac{p}{q} \leq \frac{p'}{q'}$ then H has a (p', q')-coloring.

Proof :
a) Since any $\chi(H)$-coloring of H is a $(\chi(H), 1)$-coloring of H then $\chi_c(H) \leq \chi(H)$. If $\chi_c(H) = p/q \leq \chi(H) - 1$ and c is a (p, q)-coloring of H then $c' : V(H) \longrightarrow \{1, .., \lceil \frac{p}{q} \rceil\}$ define by $c'(x) = \lceil \frac{c(x)}{q} \rceil$ is an $\lceil \frac{p}{q} \rceil$-coloring of H. Therefore $\chi(H) \leq \lceil \frac{p}{q} \rceil$, a contradiction.
b) By proposition 1 there exists homomorphism $f : H \longrightarrow G^p_q$ such that $f(x) = c(x)$. Suppose f is not onto then there exists vertex v of G^p_q such that f is a homomorphism from H to $G^p_q - v$. Since $\chi_c(G^p_q - v) < \frac{p}{q}$, by Theorem 2.a we have $\chi_c(H) < \frac{p}{q}$, and it is a contradiction.
c) This case is followed by Bondy and Hell's theorem and Proposition 1. □

Lemma 4 *Let H be a k-uniform hypergraph then*

a) $\chi_f(H) \leq \chi_c(H)$,

b) If H is vertex transitive then $\chi_f(H) = \frac{V(H)}{\alpha(H)}$.

Proof : a) Let $\chi_c(H) = \frac{p}{q}$, and c be a (p, q)-coloring of H. Define the independent set $S_i = \bigcup_{j=i}^{i+q-1} c^{-1}(j)$. The mapping f that assign to each S_i the value $\frac{1}{q}$ and assign the value 0 to other independent set is a fractional coloring which has value $\frac{p}{q}$.
b) Suppose f is a fractional coloring of H with value $\chi_f(H)$. Now we have $|V(H)| = \sum_{s \in S} |s| f(s) \leq \alpha(H)\chi_f(H)$. Let $A_1, .., A_n$ be all of the independent set of size $\alpha(H)$ and each vertex of H appears exactly in l set of them (since H is vertex transitive then there exists such l). The mapping f that assign to each A_i the value $\frac{1}{l}$ and assign the value 0 to other independent set is a fractional coloring which has value $\frac{|V(H)|}{\alpha(H)}$, and proof is complete. □

Every k-uniform complete hypergraph H is vertex transitive and $\alpha(H) = k - 1$. Therefore by Lemma 4 $\chi_f(H) = \frac{|V(H)|}{k-1}$. It is easy to see that H has a

$(|V(H)|, k-1)$-coloring; therefore, $\chi_c(H) = \frac{|V(H)|}{k-1}$.

Theorem 5 *For every p, q and k with, $2q \leq p$, there exists a k-uniform hypergraph H where $\chi_c(H) = \frac{p}{q}$.*

Proof: Let $rq \geq k$. Consider the hypergraph $H^{pr}_{qr}(k)$. It is obvious that $H^{pr}_{qr}(k)$ is vertex transitive and $\alpha(H^{pr}_{qr}(k)) = qr$; therefore, by Lemma 4 $\chi_f(H^{pr}_{qr}(k)) = \frac{p}{q} \leq \chi_c(H^{pr}_{qr}(k))$. But the mapping $f : H^{pr}_{qr}(k) \longrightarrow G^{pr}_{qr}$ by $f(x) = x$ is a homomorphism; therefore, by Theorem 2.b we have $\chi_c(H^{pr}_{qr}(k)) \leq \chi_c(G^{pr}_{qr}) = \frac{p}{q}$. □

4 Relation Between Chromatic Number And Circular Chromatic Number

In this section we introduce some sufficient conditions guarantee the equality between chromatic and circular chromatic number of a k-uniform hypergraph. Universal vertex is a sufficient condition for equality between circular chromatic number and circular chromatic number. For every k there exists a k-uniform hypergraph H with a universal vertex such that $\chi_c(H) < \chi(H)$. For example every vertex of a k-uniform complete hypergraph H is universal. But $\chi_c(H) = \frac{|V(H)|}{k-1}$ and it may not be integer. Therefore in contrary to graphs, universal vertex is not a sufficient condition for equality between chromatic number and circular chromatic number. In the following we propose some sufficient condition that guarantee the equality between chromatic number and circular chromatic number.

Theorem 6 *Suppose H is a k-uniform hypergraph , and v is a universal vertex of H*

a) If for every subset A of $V(H) - \{v\}$ with $|A| \leq 2k - 4$, we have $\chi(H - v - A) = \chi(H - v)$, Then $\chi_c(H) = \chi(H)$.

b) If $\chi(H - v) = \chi(H)$, and for every subset A of $V(H)$ with $|A| \leq 2k - 4$ we have $\chi(H - A - v) \geq \chi(H) - 1$. Then $\chi_c(H) = \chi(H)$.

Proof:
a) Let $\chi(H-v) = n$. If $\chi(H) = n$ then in any n-coloring of H the color class A, containing v, has at most $k-1$ vertices, and $\chi(H-v-A) = n-1$, contradiction. Therefore $\chi(H) = n + 1$. Let $\chi_c(H) = \frac{p}{q} < n + 1$ and c be a (p, q)-coloring of H, where $c(v) = 0$. If $q > k - 1$, by Theorem 3.b c is onto; therefore, there exist vertices $x_1, .., x_{k-1}$ of H such that $c(x_i) = i$, $1 \leq i \leq k - 1$. Now the edge $\{v, x_1, .., x_{k-1}\}$ does not satisfy the (p, q)-coloring condition. Therefore

$q \leq k-1$. Let $C_1 = c^{-1}(i)$, $0 \leq i \leq q-1$, and $C_2 = c^{-1}(i)$, $p-q+1 \leq i \leq p-1$, and $C = C_1 \bigcup C_2$. We have $|C_1| \leq k-1$ and $|C_2| < k-1$. Otherwise C_1 or $C_2 \bigcup \{v\}$ has an edge which does not satisfy (p,q)-coloring condition. The coloring $c' : V(H-v-C) \longrightarrow \{0,1,..,n-3\}$ which is define by $c'(x) = \lfloor \frac{c(x)-q}{q} \rfloor$ is an $(n-2)$-coloring of $H-v-C$, a contradiction. Therefore $\chi_c(H) = \chi(H)$.
b) The proof is similar to part a. □

In the proof of Theorem 4 we showed that if v be a universal vertex of a k-uniform hypergraph H and $\chi_c(H) = \frac{p}{q}$ then $q < k$. We use this fact in the following theorem.

Theorem 7 *Let H be a 3-uniform hypergraph and $v \in V(H)$ be a universal vertex. If $\chi(H) = \chi(H-v)$ then $\chi_c(H) = \chi(H)$.*

Proof : Let $\chi(H) = n$ and $\chi_c(H) = \frac{p}{q} < n$. Since v is a universal vertex, we have $q = 2$, and $p = 2n-1$. Let c be a $(2n-1, 2)$-coloring for H. Without loss of generality we can assume that $c(v) = 0$. Since c is onto then $|c^{-1}(\{0,1\})| = |c^{-1}(\{0, 2n-2\})| = 2$. Let $c(w) = 1$ and $c(u) = 2n-2$. The coloring c' : $V(H-u-v-w) \longrightarrow \{0,1,..,2n-5\}$ which is defined by $c'(x) = \lfloor \frac{c(x)-2}{2} \rfloor$ is an $(n-2)$-coloring of $H - \{v,u,w\}$; therefore, $\chi(H - \{u,v,w\}) \leq n-2$. Now define $f : V(H-v) \longrightarrow \{1,2,..,n-1\}$ by $f(x) = c'(x)$ if $x \notin \{u,w\}$ and $f(u) = f(w) = n-1$. Since f is an $(n-1)$-coloring of $H-v$, we have a contradiction. □

let G be a graph and $\chi(G) = n$. If there is a nontrivial subset A of $V(G)$ such that in any n-coloring c of G, each color class X of c is either contained in A or disjoint from A then $\chi_c(G) = \chi(G)$. And consequently if G has a universal vertex, or G is uniquely n-colorable then $\chi_c(G) = \chi(G)$. Similar to graphs we can prove that if for a k-uniform hypergraph H $\chi(H) = n$ and there is a nontrivial subset A of $V(H)$ such that for any n-coloring c of H, each color class X of c is either contained in A or disjoint from A, then $\chi_c(H) = \chi(H)$. By considering any color class of an uniquely n-colorable k-uniform hypergraph H as A, we have $\chi_c(H) = \chi(H)$. Now for each k and n we construct an uniquely n-colorable k-uniform hypergraph H.

Let $m \geq 2k-3$, and $X_1, X_2, \cdots X_n$ be disjoint m-sets. Let $V(H) = \bigcup_i X_i$ and k-subset e of $V(H)$ is an edge if it is not contained in X_i , $1 \leq i \leq n$. It is easy to see that $\chi(H) = n$. Now let c be a n-coloring of H, we show that for all i, $|c(X_i)| = 1$. Suppose it does not follow. Without loss of generality assume x and y are in X_1 and $c(x) = 1$, $c(y) = 2$. Now the number of vertices with color 1 or 2 in $V(H) - X_1$ is at most $k-2$ and if $i \geq 2$ then $|c^{-1}(i)| \leq m$, otherwise H has a monocolor edge. Then

$$2(k-2) + (n-2)m \geq (n-1)m.$$

Therefore $m \leq 2k-4$, contrary to our assumption. Hence H is uniquely n-colorable and $\chi_c(H) = \chi(H)$.

Now we show that for every k and n there exists k-uniform hypergraph h such that $\chi_c(H) = \chi(H) = n$.

Suppose H be a k-uniform hypergraph, and v be a vertex of H. Let A be a subset of $V(H)$ containing v such that every k-subset of A is an edge of H. Let H' be obtained from H by deleting all k-subset of A containing v. Similar to the proof of Theorem 1 in [6] we have $\chi_c(H') \geq \chi(H) - 1$. Let H be a n-colorable k-uniform hypergraph such that for edge e of H, $\chi(H-e) < \chi(H)$. Let $A = e$ and $H' = H - e$, then we have $\chi_c(H-e) \geq \chi(H) - 1 = \chi(H-e)$. Therefore $\chi_c(H-e) = \chi(H-e) = n-1$.

It was shown in [1] that for every k, l and m there exists a k-uniform hypergraph H with $girth(H) \geq l$ and $\chi(H) \geq m$; therefore, by above discussion for every n and k, there exists a k- uniform hypergraph H with girth at least l and $\chi_c(H) = \chi(H) = n$.

The Stiner Triple System are specific type of 3-uniform hypergraphs. We are looking for some Stiner Triple System of order $6n+3$, STS(6n+3), guarantee of the equality circular chromatic number and chromatic number.
Let H be a $STS(6n+3)$ constructed as follow:
$V(H) = \{a_1, a_2, .., a_{2n+1}, b_1, b_2, .., b_{2n+1}, c_1, c_2, .., c_{2n+1}\}$, and the blocks are of two types.
Type 1:.
$\{\{a_i, b_i, c_i\} \mid 1 \leq i \leq 2n+1\}$
Type 2:
$\{\{a_i, a_j, b_{i\circ j}\} \mid 1 \leq i < j \leq 2n+1\}$,
$\{\{b_i, b_j, c_{i\circ j}\} \mid 1 \leq i < j \leq 2n+1\}$,
$\{\{c_i, c_j, a_{i\circ j}\} \mid 1 \leq i < j \leq 2n+1\}$,
Where,

$$ioj = \begin{cases} \frac{i+j}{2} & \text{if } i+j \ is \ even \\ \frac{i+j-2n-1}{2} & \text{if } i+j > 2n+1 \ and \ i+j \ is \ odd \\ \frac{i+j+2n+1}{2} & \text{if } i+j \leq 2n+1 \ and \ i+j \ is \ odd. \end{cases}$$

This construction is known as Bose construction.

Theorem 8 *Suppose H is a $STS(6n+3)$ with vertex set, $V(H) = \{a_1, a_2, .., a_{2n+1}, b_1, b_2, .., b_{2n+1}, c_1, c_2, .., c_{2n+1}\}$ and H' is a $STS(6m+3)$ with vertex set$\{x_1, x_2, .., x_{2m+1}, y_1, y_2, .., y_{2m+1}, z_1, z_2, .., z_{2m+1}\}$, H and H' have Bose construction. Let $2n+1 | 2m+1$, and if $\chi_c(H) = 3$ then $\chi_c(H') = 3$.*

Proof: It is easy to see that $\chi(H) = \chi(H') = 3$. Let $(2n+1)q = 2m+1$. Consider the mapping $f : H \longrightarrow H'$ by $f(a_i) = x_{qi}$, $f(b_i) = y_{qi}$, $f(c_i) = z_{qi}$. It is obvious that $qi \circ qj = q(i \circ j)$. Therefore f is a homomorphism, and by Theorem 2.b, $\chi_c(H) \leq \chi_c(H')$, and hence $\chi_c(H') = 3$. □

With computer research we found out that the circular chromatic number of a STS(15) and STS(21), constructed by Bose construction, is 3. Therefore if $5 | 6n+3$ or $7 | 6n+3$ then every STS(6n+3) by Bose construction has circular chromatic number 3.

We know that for every induced subhypergraph H' of a k-uniform hypergraph H, $\chi_c(H') \leq \chi_c(H)$. Therefore for every vertex v of H, $\chi_c(H-v) \leq \chi_c(H)$. The question may be raised is that how much does deleting a vertex decrease the circular chromatic number? By Theorem 3.a we have $\chi_c(H) - \chi_c(H-v) < 2$. The following theorem shows that this difference can be very close to 2.

Theorem 9 *For every k and for every real number $\epsilon > 0$ there exists a k-uniform hypergraph H and a vertex v of H such that $\chi_c(H)-\chi_c(H-v) > 2-\epsilon$.*

Proof: Consider $n \in N$ such that $\frac{1}{n} < \epsilon$. Let $H' = H_n^{nr+1}(k)$, and $H_1, H_2, .., H_{2k-3}$ be $2k-3$ disjoint copy of H'. Let H be a k-uniform hypergraph , and $v \in V(H)$ be a universal vertex, such that $V(H-v) = \bigcup_{1\leq i\leq 2k-3} V(H_i)$ and $E(H-v) = \bigcup_{1\leq i\leq 2k-3} E(H_i)$. H satisfies conditions of Theorem 4; therefore, $\chi_c(H) = \chi(H) = r+2$. Since $\chi_c(H-v) = r+\frac{1}{n}$, then $\chi_c(H)-\chi_c(H-v) = 2-\frac{1}{n} > 2-\epsilon$.□

Theorem 10 *Let H be a k-uniform hypergraph H, $\chi(H) = n$, and v be a universal vertex of H, then there exists vertex u of H such that $\chi_c(H-u) \geq \chi(H) - 1$.*

Proof : If there exists a vertex u such that $\chi(H-u) = \chi(H)$ then u is our desired vertex. Let for every vertex u, $\chi(H-u) = n-1$. Therefore, in every $(n-1)$-coloring of $H-u$ every color class has at least $k-1$ vertices. Let $u \neq v$, and $\chi_c(H-u) = \frac{p'}{q'}$, and $\chi_c(H) = \frac{p}{q}$, where $q < k$ and $q' < k$. If $\chi_c(H-u) = n-1$ then u is the desired vertex, otherwise let $p' = (n-2)q' + \beta$ where $0 < \beta \leq q'-1 \leq k-2$. Let f be a (p', q')-coloring of $H-u$, and $f(v) = 0$. Consider $A = \{x \mid p'-\beta \leq f(x) \leq p'-1 \}$. If $|A| \geq k-1$, and $x_1, .., x_{k-1} \in A$, then the edge $\{v, x_1, .., x_{k-1}\}$ does not satisfy (p', q')-coloring condition. Therefore, $|A| \leq k-2$. Define $(n-1)$-coloring c of $H-u$ by $c(x) = \lfloor \frac{f(x)}{q'} \rfloor$, and consider $c^{-1}(n-2)$. It can be seen that $c^{-1}(n-2) = A$, and since $|A| \leq k-2$, we have $(n-1)$-coloring, c, of $H-v$ such that c has a color class, A, with size at most $k-2$. It is a contradiction. □

According to the above theorems the following conjecture and question seems to be natural.

Conjecture 1 Let H be a k-uniform hypergraph. Then H has a vertex u such that $\chi_c(H-u) \geq \chi_c(H) - 1$.

Question 2 Let H be a k-uniform hypergraph. Is it true that H has a vertex u such that $\chi_c(H-u) \geq \chi_c(H) - \frac{1}{k-1}$?

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