1. Answer the following short questions.

(a) Is an element of a set a special case of a subset of a set?
   An element of \( A \) cannot be an element of \( \mathcal{P}(A) \).

(b) List all subsets of \( \{a, b, c\} \) containing \( a \) but not containing \( b \).
   The subsets are \( \emptyset, \{a\}, \{a, c\} \).

(c) What is the intersection of the set of positive integers whose last digit is 3, and the set of even integer.
   Empty set.

(d) What is the intersection of the set on integers divisible by 5 and the set of even integers.
   Empty set

(e) Let \( A = \{a, b, c, d, e\} \) and \( B = \{c, d, e\} \). List all subsets of \( A \) whose intersection with \( B \) has 1 element.
   The subsets are \( \{c\}, \{d\}, \{e\}, \{a, c\}, \{b, c\}, \{a, d\}, \{b, d\}, \{a, e\}, \{b, e\}, \{a, b, c\}, \{a, b, d\}, \{a, b, e\} \).

(f) Three sets have 5, 10, and 15 elements. How many elements can their union and intersection have.
   Minimum count: Union 15 (the third set contains the other two); Intersection 0 (the sets may not have any element in common).
   Maximum count: Union 30, Intersection 5.

(g) What is the symmetric difference of \( A \) and \( A \)?
   Empty set

(h) What is the difference \( A - B \) if \( A \) is the set of primes and \( B \) is the set of positive odd integers?
   The set \( \{2\} \).

(i) What is the difference \( A - B \) if \( A \) is the set of nonnegative integers and \( B \) is the set of nonpositive real numbers.
   The set \( \{1, 2, 3, \ldots\} \)

2. () Write the following set by listing their elements between braces.

(a) \( \{x \in \mathbb{Z} : -3 < x \leq 2\} \)
   \( \{-2, -1, 0, 1, 2\} \)

(b) \( \{x \in \mathbb{R} : \sin x = 0\} \)
   \( \{\ldots, -2\pi, \pi, 0, \pi, 2\pi, \ldots\} \)

(c) \( \{x \in \mathbb{R} : \sin \pi x = 0\} \)
   \( \{\ldots, -2, -1, 0, 1, 2, \ldots\} \)
(d) \( \{ x \in \mathbb{R} : x^2 = 7 \} \)
\( \{-\sqrt{7}, \sqrt{7}\} \)

(e) \( \{ x \in \mathbb{Z} : 2x < 5 \} \)
\( \{-2, -1, 0, 1, 2\} \)

(f) \( \{ x \in \mathbb{Z} : -2 < x \leq 7 \} \)
\( \{-1, 0, 1, 2, 3, 4, 5, 6, 7\} \)

(g) \( \{ x \in \mathbb{Z} : -5 < x \leq 2 \} \)
\( \{-4, -3, -2, -1, 0, 1, 2\} \)

3. () Write the following set in set-builder notation.

(a) \( \{ 0, 1, 4, 9, \ldots \} \)
\( \{ x^2 : x \in \mathbb{Z} \} \)

(b) \( \{ 2, 3, 5, 7, 11, \ldots \} \)
\( \{ x : x \text{ is a prime number} \} \)

(c) \( \{ 3, 4, 5, 6, 7, 8 \} \)
\( \{ x : -3 \leq x \leq 8 \} \)

(d) \( \{ -3, -2, -1, 0, 1, 2, 3 \} \)
\( \{ x : -3 \leq x \leq 3 \text{ and } x \neq 1 \} \)

(e) \( \{ 0, 1, 8, 27, 64, 125, \ldots \} \)
\( \{ x^3 : x \in \mathbb{Z}^+ \} \)

(f) \( \{ 0, -1, -4, -9, \ldots \} \)
\( \{-x^2 : x \in \mathbb{Z}\} \)

4. () Let \( \mathbb{R} \) be the universal set. Let \( A = \{ 1 \} \), \( B = (0, 1) = \{ x : 0 < x < 1 \} \)
and \( C = [0, 1] = \{ x : 0 \leq x \leq 1 \} \). Write down the following sets.

- \( A \cup B \)
- \( A \cap B \)
- \( B \cap C \)
- \( A \cup C \)
- \( A \cap C \)

\( (0,1) \quad \phi \quad (0,1) \quad [0,1] \quad \{1\} \)

Are any of the pairs of sets \( A, B \) and \( C \) disjoint? \( A \) and \( B \) are disjoint.

5. () Let \( A, B \) and \( C \) be three arbitrary subsets of the universal set \( U \). Use an element containment proof (i.e. prove that the left side is a subset of the right side and the right side is a subset of the left side) to prove the following:

- \( \overline{A \cup B} = \overline{A} \cap \overline{B} \)

Let \( x \in \overline{A \cup B} \)
\( \Rightarrow x \notin A \cup B \)
\( \Rightarrow x \notin A \) and \( x \notin B \)
\( \Rightarrow x \in \overline{A} \) and \( x \in \overline{B} \)
\( \Rightarrow x \in \overline{A} \cap \overline{B} \)
So \( \overline{A \cup B} \subseteq \overline{A} \cap \overline{B} \).
Let \( x \in A \cap B \)  
\[ \Rightarrow x \in A \text{ and } x \in B \]  
\[ \Rightarrow x \notin A \text{ or } x \notin B \]  
\[ \Rightarrow x \in A \cup B \]  
So \( A \cap B \subset A \cup B \)  
Since \( A \cap B \) and \( A \cup B \) are both subsets of one another, so \( A \cup B = \bar{A} \cap B \).

- \( A \cap B = A \cup B \).
- \( A \cup B \cap C = A \cap B \cap C \).
- \( A \cap B \cap C = A \cup B \cap C \).
- \( A \cap B \cap C = A \cup B \cup C \).  
Proved in the same way as above.

6. Prove that \(|A \cup B| + |A \cap B| = |A| + |B|\).

7. Use the membership table method to determine which membership \( \subseteq, =, \supseteq \) is true for the following pair of sets.

- \((A - B) \cup (A - C), A - (B \cap C)\).

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From above, we see that \( A - (B \cap C) \) and \( (A - C) \cup (B - C) \) are equal.

- \((A - C) - (B - C), (B - A)\).

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From above table we see that \( (A-C)-(B-C) \) is the subset of \( (A-B) \).

- \((B - C), (B - A) - (C - A)\).

8. (Bonus) (10 points) Prove that \( A \times (B \cap C) = (A \times B) \cap (A \times C) \).

Let \((x, y) \in A \times (B \cap C)\)
11. (Bonus Question) A random experiment consists of rolling an unfair, six-sided die. The digit 6 is three times as likely to appear as the numbers 2 and 4. The numbers 2 and 4 are twice as likely to appear as one of the numbers, 1, 3, and 5.

Assign appropriate probabilities to the six outcomes in the sample space.

9. Let $n$ be an odd integer. Prove that $n^3 + 2n^2$ is also odd.

Let $p : n$ is an odd integer; $q : n^3 + 2n^2$ is odd. We need to show $p \rightarrow q$.

(a) Use a direct proof.

Let odd $n = 2t + 1$, $t$ an integer. Then $(2t + 1)^3 + 2(2t + 1)^2 = (8t^3 + 12t^2 + 6t + 1) + 2(4t^2 + 4t + 1) = 2(4t^3 + 10t^2 + 7t + 1) + 1$. This implies that $q$ is true.

(b) Use an indirect proof.

We need to show that $\neg q \rightarrow \neg p$, i.e., we need to show that $n^3 + 2n^2$ is even implies $n$ is even. Since $2n^2$ is even, $n^3 + 2n^2$ is even implies $n$ is even.

(c) Use a proof by contradiction.

Here we assume that $p$ is true and $\neg q$ is true. $n^3 + 2n^2$ is even implies that $n^3$ is even. Therefore, $n$ is even, i.e. $\neg p$ is true. Thus we have $p$ is true and $\neg p$ is true. We arrive at a contradiction. Therefore, our assumption the $\neg q$ is true is false. Therefore $p \rightarrow q$ is true.

10. Two fair six-sided dice are rolled and the sum $s$ of the numbers coming up is recorded. What is the probability of $s \geq 10$? Show your work.

Let $S$ be the sample space of the experiment of throwing two dice (red and blue). Then $S = \{(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6), \ldots, (6,11), (6,2), \ldots, (6,6)\}$. The cardinality of $S$ is 36. The probability of an outcome is $\frac{1}{36}$. Let $A \subseteq S$ where $A = \{(a,b) \in S|a + b \geq 10\}$. Hence $A = \{(4,6), (5,5), (6,4)\}$. Therefore, $Pr(A) = \frac{|A|}{|S|} = \frac{3}{36}$.

11. (Bonus Question) A random experiment consists of rolling an unfair, six-sided die. The digit 6 is three times as likely to appear as the numbers 2 and 4. The numbers 2 and 4 are twice as likely to appear as one of the numbers, 1, 3, and 5.

Assign appropriate probabilities to the six outcomes in the sample space.
Since the die is unfair, the probability of the outcomes are not the same. Let $p$ be the probability of each of the outcomes of 1, 3 and 5. Then the probability of each of the outcomes of 2 and 4 is $2p$. The probability of outcome 6 is $6p$. Since $6p + 4p + p + p + p = 1$, therefore $p = \frac{1}{13}$. Therefore, $Pr(\{1\})Pr(\{3\}) = Pr(\{5\}) = 1/13$. $Pr(\{2\}) = Pr(\{4\}) = \frac{2}{13}$. Lastly, $Pr(\{6\}) = \frac{6}{13}$. 