MACM 101 : Quiz 5 (Version 3): (March 27, 2018; during the tutorial) (Full Marks 40) Time: 50 minutes

Please read the questions carefully. Return this paper along with the answer sheet. Credit will be assigned based on the clarity of the arguments. Explain your answers and provide as much detail as possible.

Each question is worth 5 points.

1. Let \( A = \{6 : 00, 6 : 30, 7 : 00, 7 : 30, \ldots, 9 : 30, 10 : 00\}\) denote the set of nine half-hour periods in the evening. Let \( B = \{3, 12, 15, 17\}\) denote the set of four local television channels. Let \( R_1 \) and \( R_2 \) be two relations from \( A \) to \( B \), \( R_1, R_2 \subseteq A \times B \). Give examples of \( R_1 \) and \( R_2 \) such that \( |R_1| = 4 \) and \( |R_2| = 4 \) and \( |R_1 - R_2| = 1 \).

**Solution:** \( R_1 = \{(6 : 30, 3), (7 : 00, 12), (7 : 30, 15), (8 : 00, 17)\} \) and \( R_2 = \{(6 : 30, 3), (7 : 00, 12), (7 : 30, 15), (10 : 00, 3)\} \). In this case, \( R_1 - R_2 = \{(8 : 00, 17)\} \).

2. Let \( I \) be the set of integers from 1 to 7.

   (a) Is there a natural way to interpret the ordered pairs in \( I \times I \) as points in the plane?

   **Solution:** In the plane \( I \times I \) represents a \( 7 \times 7 \) grid. The coordinate of each grid point is \((i, j), 1 \leq i \leq 7, 1 \leq j \leq 7\).

   (b) What are the elements of the relation \( R \subseteq I \times I \) defined by \( xRy \iff x \leq y \)? Give a matrix representation of the relation \( R \).

   **Solution:** The elements of \( R = \{(1,1),(1,2),\ldots,(1,7),(2,2),(2,3),\ldots,(2,7),\ldots(7,7)\} \).

   Matrix representation: The lower triangular elements and the diagonal elements are all 1s. The upper triangular elements are all 0s.

   (c) Let \( R' \) be a binary relation on \( I \times I \), i.e. \( R' \subseteq (I \times I) \times (I \times I) \).

      - Write an element of \( R' \). **Solution:** \(( (1,5), (3,7) ) \) \( \in ( (I \times I) \times (I \times I) ) \).

      - What is the cardinality of the set \(( I \times I ) \times ( I \times I ) \)? **Solution:** 49 \times 49.

3. If \( B = \{a, b, c, d, e\} \) and \( R \) is an equivalence relation on \( B \) which has the partition \( \{a, b, c\} \) and \( \{d, e\} \). Draw the directed graph representing \( R \).

   **Solution:** Each pair \( x, y \) of the elements of each partition generates directed edges \((x,y)\) and \((y,x)\). For each element \( x \), there is a directed edge \((x,x)\).

4. Relation \( R \) is given by the following matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Is \( R \) a partial order? If yes, justify; what its minimal, maximal, least, and greatest elements are; how does the Hasse diagram look like?

**Solution:** Let the relation is defined on a set \( A = \{1, 2, 3, 4\} \). Now the relation can be written as \( R = \{(1,1),(2,2),(3,3),(4,4),(3,1),(3,2),(4,1),(4,2),(4,3)\} \). It is a partial order. It is reflexive, anti-symmetric and transitive. The Hasse diagram looks like:
1 and 2 are maximal elements; 4 is the least element. There is no greatest element.

5. Let $R$ be a relation on the set of propositions such that $R = \{(p, q) | p \leftrightarrow q \text{ is true}\}$. Determine whether $R$ is reflexive, symmetric, antisymmetric, transitive, partial order, and equivalence relation. Just Yes/No answer is fine.

Solution: $R$ is a reflexive relation since any proposition is equivalent to itself. $R$ is clearly symmetric and also transitive.

6. For $|A| = 5$, how many relations $R$ on $A$ are there? How many of these relations are symmetric?

Solution: Any relation on $A$ can be represented by a $5 \times 5$ matrix. Since the matrix to be set has to be symmetric, (see figure) if we set all the * locations, all the + locations in the matrix are automatically selected due to its symmetric property. We thus need to set $5 + 4 + 3 + 2 + 1 = 15$ locations. Therefore, there are $2^{15}$ different ways to construct symmetric relations on a set of five elements.

7. The Fibonacci sequence is defined as the sequence starting with $F_1 = 1$ and $F_2 = 1$ and then for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

(a) Determine the values of $F_3$, $F_4$, $F_5$, $F_6$ and $F_7$. Solution: 2, 3, 5, 8, 13.

(b) Prove by induction that $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$.

Solution: Let $S(n)$ be the statement $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ for $n \geq 1$.

Basis: Clearly, $S(1), S(2)$ are true.

Inductive Step: We want to show that $S(1) \land S(2) \land \ldots \land S(k) \rightarrow S(k+1)$ for $k \geq 2$.

We need to show that $F_1 + F_2 + \ldots + F_k + F_{k+1} = F_{k+3} - 1$.

LHS = $F_1 + F_2 + \ldots + F_k + F_{k+1} = (F_1 + F_2 + \ldots + F_k) + F_{k+1} = (F_{k+2} - 1) + F_{k+1}$ (since $S(k)$ is true).
Therefore, LHS = \((F_{k+2} + F_{k+1}) - 1 = F_{k+3} - 1\). (From the definition of \(F_{k+3}\)).
Thus \(S(k+1)\) is true. Therefore, by the principle of strong induction we can conclude that \(S(n)\) is true for all \(n \geq 1\).

8. Prove by induction the following generalization of De Morgan's law to \(n \geq 1\) sets:
\[
\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}.
\]

**Solution:** This is proved in the lecture slides.