Spectral methods

- ... use eigenstructures of appropriately defined linear mesh operators to solve a problem

- Eigenstructures:
  - Eigenvalues
  - Eigenvectors
  - Projections
    - into an eigensubspace, or
    - along eigenvectors

\[ Au = \lambda u, \quad u \neq 0 \]

\[ A = U\Lambda U^T \]

\[ y' = U_{(k)} U_{(k)}^T y \]

\[ \hat{y} = U^T y \]

Outline

- Two looks on spectral methods
- Overall algorithmic structure
- Application examples
- Why spectral methods for mesh processing?
- Difficulties and open questions
- Conclusion

Much more details in ...

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What do you see?

From another angle

A solution paradigm

- Viewing a problem from a different angle …
- Problem solving in an alternative domain — the spectral domain
- A transform-based technique: e.g., geometric dual transform, Fourier transform, Hough transform, etc.
From a geometric perspective

- Euclid’s classical geometry
  - Primitives not represented by numbers
  - Geometric relationships deduced in a pure and self-contained manner, via axioms
- Descartes’ analytic geometry
  - Algebraic and analysis tools introduced
  - Primitives referenced by a global frame — extrinsic approach

Intrinsic geometry

- Riemann’s intrinsic view of geometry
  - Geometric relationships specified by a “metric”
  - Metric: “distance” between points on surface
  - Many spaces can be treated simultaneously — isometries
- The spectral approach takes this view
  - Intrinsic mesh information captured via an operator
  - Eigenstructures effectively characterizes intrinsic geometry

Overall structure

Input mesh $y$

Eigendecomposition: $A = U \Lambda U^T$

Some mesh operator $A$

E.g. $y^* = U^{(3)} U^{(3)T} y$

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Ex. 1: Spectral mesh transform

- **Operator:** the graph Laplacian
  \[ K = D - W \]
  
  $W$: adjacency matrix of mesh graph and $D$: diagonal matrix of $W$’s row sums. The

- Eigendecomposition: $K = U \Lambda U^T$

- Mesh $y$ as linear sum of eigenvectors:
  \[ y = U \Lambda \] or \[ y = u_1 \tilde{y}_1 + u_2 \tilde{y}_2 + \cdots + u_n \tilde{y}_n \]

- $\tilde{y} = U^T y$: a spectral transform of $y$ with respect to $K$ — "Fourier-like"

Applications

- Eigenvectors organized by ascending eigenvalues
- JPEG-like geometry compression [Karni & Gotsman 00]: reconstruct mesh geometry by **truncated spectrum**

\[ y'(k) = u_1 \tilde{y}_1 + u_2 \tilde{y}_2 + \cdots + u_k \tilde{y}_k \]

- Mesh filtering, e.g., [Vallet & Levy 07], or watermarking [Obuchi et al. 01]

Example 2: spectral clustering

Encode information about pairwise point affinities

Input data \[ A_j = e^{-\frac{||p_j - p||^2}{2\sigma^2}} \]

Operator \( A \)

Spectral embedding

Leading eigenvectors

Spectral clustering continued

In spectral domain

Perform any clustering (e.g., k-means) in spectral domain
Applications

- Whenever clustering is applicable, e.g.,
  - **Mesh segmentation** in spectral domain [Liu & Zhang 04, 05, 07]
  - **Surface reconstruction**: grouping “inside” and “outside” tetrahedra [Kullori et al. 05]
  - **Shape correspondence**: finding clusters of consistent or agreeable pairwise matchings [Leordeanu & Hebert 06]

Ex. 3: Eigenvalues for retrieval

- **Operator**: Matrix of Gaussian-filtered pair-wise geodesic distances between 20 samples
- Farthest point sampling on dense mesh
- **Shape descriptor (EVD)**: eigenvalues $\lambda_1, \ldots, \lambda_{20}$ of $20 \times 20$ matrix [Jain & Zhang 06]
- Use $\chi^2$ distance between EVD’s for retrieval

$$\text{Dist}_{EVD}(P, Q) = \frac{1}{2} \sum_{i=1}^{20} \frac{\left| \lambda_i^P - \lambda_i^Q \right|^2}{\lambda_i^P + \lambda_i^Q}.$$ 

Some results

- LFD: Light-field descriptor [Chen et al. 03]
- SHD: Spherical Harmonics descriptor [Kazhdan et al. 03]

Bending removed …

- **Operator**: defined by geodesics again
New results

LFD-S: LFD on spectral embedding
SHD-S: SHD on spectral embedding

Spectral mesh correspondence

Geodesic-based spectral embedding
Eigenvector scaling
Non-rigid ICP via thin-plate splines

Best matching

[Jain & Zhang 06]

Related developments

- Spectral embedding via multidimensional scaling and retrieval via moment shape descriptors [Elad & Kimmel 03]
- Laplacian-Beltrami operator
  - Spectra: shape DNA [Reuter et al. 06]
  - Shape distributions: [Rustamov 07]
- Graph indexing [Sundar et al. 03, …]
- Many applications using spectral embeddings (1D, 2D, 3D, higher-D); see [Zhang et al. 07]

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Why spectral?

- Spectral analysis via a mesh Laplacian allows for a Fourier-like analysis for meshes
- Capturing of global and intrinsic shape characteristics
- Dimensionality reduction

Classical 1D DFT

- Discrete Fourier transform ⇔ decomposition of signal into linear sum of eigenvectors of the 1D discrete Laplacian
- Fixed set of Fourier basis

The mesh case

- Classical 1D Laplacian replaced by some mesh Laplacian (different choices)
- Eigenvectors of mesh Laplacian resemble behavior of DFT basis
- Basis changes depending on mesh operator

The mesh case, continued

- Eigenvalue ⇔ frequency and eigenvector ⇔ vibration mode [Taubin 95]
  - Vibration pattern: non-regular
  - Frequency ⇔ total energy: only a global characterization
  - Very little known is about geometric behavior of Laplacian eigenfunctions
Modeling of global characteristic

- Eigenvalues
  - [Chung 97]: Graph spectra closely related to almost all major graph invariants, e.g., diameter, connectivity, etc.
  - Graph or mesh spectra adopted as compact global shape descriptors [Shokoufhandeh et al. 05, Reuter et al. 06, Jain & Zhang 06, …]
- Eigenvectors
  - Much refined shape information
  - E.g., extremal properties (Courant-Fisher Theorem): heuristic for NP-hard problems, e.g., normalized cuts for segmentation [Shi & Malik 00]

Example: clustering again

- Linkage-based (local info.)
- Spectral clustering

Local vs. global distances

- Points in same cluster closer to each other in transformed domain than points in different clusters
- Instead of a single shortest path, look at set of shortest paths — more global!
- Commute time distance $c_{ij} =$ expected time it takes a random walk to go from $i$ to $j$ and then back to $i$

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Commute time, spectral embedding

- More specifically, eigendecompose the graph Laplacian
  \[ K = U \Lambda U^T \]

- Let \( K' \) be the \textbf{generalized inverse} of \( K \),
  \[ K' = U \Lambda' U^T, \]
  \( \Lambda'_{ii} = 1/\Lambda_{ii} \) if \( \Lambda_{ii} \neq 0 \), otherwise \( \Lambda'_{ii} = 0 \).

- Let \( z_i \) be the \( i \)-th row of \( U \Lambda^{1/2} \) — this is the spectral embedding

- Then
  \[ ||z_i - z_j||^2 = c_{ij} \]
  the commute time distance [Klein & Randic 93, Fouss et al. 06]

Intrinsic shape characteristic

- Bending-invariant embedding [Elad & Kimmel 03, Jain & Zhang 07]

Other example: clustering again

- Non-Gaussian “clusters”

Example: Results of \( k \)-means

- Clusters of Gaussians
- Non-Gaussian “clusters”
Example: Spectral embedding

Original rings data Spectral embedding in 2D

Clustering more pronounced!

Example: Spectral clustering

- Maps nonlinear structures into a high-dimensional space
- Non-linearities “unfolded” into linear ones to enable PCA, $k$-means, …

Isomap: [Tenenbaum et al. 00]

Dimensionality reduction

- Feature space mapped into a high-dimensional space
- Use few dominant eigenvectors or eigenvalues for representation — dimensionality reduction
  - Information-preserving — Eckart & Young’s theorem on low-rank approximations
  - Enhancement of structures — Polarization theorem
  - Low-dimensional representation: simplifying solutions

Example: eigenspace projection

- Mesh signal projected into the eigenspace formed by the first two eigenvectors of a mesh Laplacian [Liu & Zhang 07]
- Shape analysis on 3D meshes reduced to contour analysis
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Eigenmode switching problem

- Many applications but approaches still rather ad-hoc
- Little is known about geometric behavior of Laplacian eigenfunctions
  - Critical points, nodal sets, $L^p$ norm, symmetry property, etc.
- Eigenvalues not sufficient to characterize eigenvectors
  - E.g., unreliable to sort eigenmodes by magnitude of eigenvalues — eigenmode switching

Difficulties and open questions

- Which operator to use?
  - Laplacian vs. higher-order operator?
  - Which Laplacian?
  - “No free lunch!” [Wardetzky et al. 07]
  - Use “nice” meshes, e.g., Delaunay [Dyer et al. 07] or non-obtuse [Li & Zhang 06]
- Sensitivity to topological changes
- Computational issues

Nice meshes

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Conclusions

- Transforming problem into spectral domain may facilitate solution to the problem
- Mesh to operator: extraction of intrinsic mesh information but dimensionality lifted
- Eigenstructures give characterization of mesh geometry in an organized and compact manner
- Though eigenstructures understand geometry [Levy 06], there is much we do not know about them!

Thank you!