Delaunay Meshes

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Delaunay structures in 2D
Voronoi diagram
Voronoi-Delaunay dual
Delaunay triangulation (DT)

Books on DT: [Edelsbrunner 01], [Cheng & Dey 09]
Defining property 1: angle sum

Opposite angle sum $\leq \pi$
Defining property 1: angle sum

Opposite angle sum $\leq \pi$
Defining property 2: empty circle

Empty circumcircle
Defining property 2: empty circle

Empty circumcircle

etc.
Degenerate case: co-circularity
Degenerate case: co-circularity
Degenerate case: quadrilateral in dual
Construction via edge flipping

- NLD (non-locally Delaunay) edge ⇔ opposite angle sum > $\pi$
Construction via edge flipping

- NLD (non-locally Delaunay) edge $\iff$ opposite angle sum $> \pi$
Any NLD edge is flippable

- In the plane, un-flippable edge leads to a “fold”
Extremal properties

- Maximization of minimum angle
- Minimization of total harmonic index [Musin 1997]

Harmonic index $= \frac{a^2 + b^2 + c^2}{\text{Area}}$

- Help prove termination of edge flipping
DT summary

- Pretty good triangulations
  - Efficiently constructed (cf. minimum weight triangulation)
  - Many applications, e.g., interpolation of height field — minimum roughness [Rippa 1990]
- Arguably the most studied triangulation
- But confined to Euclidean space
How to extend to **surface** setting?
Continuous setting – Voronoi

An intrinsic Voronoi diagram
Voronoi-Delaunay dual

An intrinsic DT or iDT
Discrete mesh setting?

An iDT mesh
Underlying surface unknown

In the absence of a surface, what is a “Delaunay” mesh?
Delaunay mesh (DM)

- DT in the discrete mesh setting
- Definition independent of underlying surface
  - In classic setting, triangulate a fixed domain
  - Now seek a “good mesh” given a point set

Delaunay meshes are pretty good meshes!
Simple definition

- A triangle mesh is a Delaunay mesh if each sum of opposite angles is \( \leq \pi \).
- Focus on closed meshes now

\[ \alpha + \beta < \pi \]

[Dyer, Zhang & Möller 07]
Why are Delaunay meshes “good”?

- Delaunay mesh is our **lunch voucher**
  - “No free lunch” paper [Wardetzky et al. 07]
  - Give “perfect” discrete Laplacians: symmetry, locality, linear precision, and positive weights
- Valid discrete harmonic maps

\[
\cot \alpha + \cot \beta = \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}
\]
Why Delaunay meshes good?

- Low-dilation graphs

\[
\text{dilation} = \max \frac{\|g(p, q)\|}{\|p - q\|}
\]
Why DM good: Low-dilation graphs

- Delaunay $\Rightarrow$ dilation $\leq 2.42$ [Keil & Gutwin 92]
- Important in network design [Eppstein 96], e.g.,
  sensor networks [Newcome & Song 03]
Even better: non-obtuse meshes

- Every angle is less than or equal to $\pi/2$

[Li & Zhang 06]
Non-obtuse meshes

- Any non-obtuse mesh is Delaunay
- Valid quasi-harmonic maps [Zayer et al. 05]
- Dilation conjectured to be 1.414
- More efficient collusion computation — point to triangle distance
Non-obtuse remeshing

- a.k.a. well-centered triangulations [VanderZee et al. 07]
- Major challenge is to get a **guarantee**
Non-obtuse meshes: modified MC

Always choose centers and midpoints
Are Delaunay meshes new?

- Restricted DT (rDT) of a point set
  - rDT needs underlying surface
  - \( \text{rDT} \neq \text{Delaunay mesh} \)

- iDT of polyhedral surface [Bobenko & Springborn 05]
  - Not a mesh in classical sense
  - Obstruct certain locality [Wardetzky 07]

- iDT mesh: edges straightened
  - \( \text{iDT} \neq \text{Delaunay mesh} \)
Delaunay mesh properties

- Empty circle? — yes, intrinsically!
- Maximization of minimum angle? — no!
- Minimization of total harmonic index? — no!
What extremal property then?

- **Area minimization**
  - A flip that reduces the sum of opposite angles also reduces total triangle area [Dyer, Zhang & Möller 07]
Area minimization

- By Bretschneider’s formula from 1842

\[
\text{Area}(ABCD) = \sqrt{(s - a)(s - b)(s - c)(s - d) - 2abcd \cos^2\left(\frac{A + C}{2}\right)}
\]

where \( s = \frac{a + b + c + d}{2} \)

If \( A + C > B + D \), then \( \cos^2\left(\frac{A + C}{2}\right) < \cos^2\left(\frac{B + D}{2}\right) \)

utilizing the fact: \( A + B + C + D \leq 2\pi \)
Degeneracy in area minimization

- Recall the degenerate situation in 2D DT:
  - Four points are co-circular, or
  - Opposite angle sums are the same
- What happens in the mesh case?
  - Opposite angles sums are the same
  - Note: both edges in a local “flip tet” can be Delaunay (total angle sum $\leq 2\pi$)
Angle sum vs. co-circularity

- There is co-circularity, but in a different domain!

Sum of opposite angles are the same

Gaussian images of four faces are co-circular
Example
DM construction via edge flipping

- Obstruction: unflippable edges — non-manifold

2-exposed case

3-exposed case
DM construction: edge splitting

- Only flip within a plane (newly inserted edges)
- Edge split near midpoint
- Split point selection via “power of 2 trick” [Ruppert 95]
- Delaunay remeshing w/o changing geometry
- Termination proof but no known complexity bound
DM construction: decimation

- via (Delaunay) constrained optimization

[Dyer, Zhang & Möller 07]
What is the deal?

- Initially, extend DT to the surface setting
- The ultimate question

Given a point set which sufficiently sample an unknown smooth surface, what is a “perfect” mesh on such a point set (vertices)

- Non-obtuse meshes or minimum (surface) area meshes [O’Rourke 1981] — No known constructions
What have we got?

- Strong indication
- Delaunay meshes are pretty good meshes
- Delaunay mesh is different from rDT and iDT
- Some basic properties, e.g., area minimization
- A construction algorithm (no size bounds) and Delaunay meshes at different level-of-details
Understanding far from enough

- How to construct a DM exclusively on a given set of points? Does it even exist? *Ideally, just flip!*
- What are the implications of un-flippable edges:
  - Un-flippable edges are due to *under-sampling*

Is a Delaunay mesh always a good mesh?
Need extra requirement

- DM can be bad!
Smooth Delaunay meshes

- Smoothness: small angles between any two normals at faces incident to a vertex
  - $\pi/2$ angle bound $\Rightarrow$ no un-flippable edges
  - Open question: DM flip $\Rightarrow$ smoothness?
Concluding remarks

- Smooth Delaunay meshes are candidates for the “ultimate” mesh
- They make Laplacian operators “perfect”
- Correlations with other well-known structures, rDT, iDT, Gabriel meshes [Dyer et al. 09]
- Our understanding far from complete
- **Faith: there is something right about them!**