Assignment #2 Solution

[5 marks] Problem 1: Valence of vertices in a triangle mesh

The well-known Euler-Poincare formula relates the number of faces \( F \), edges \( E \), and vertices \( V \) of a closed triangle mesh: \( V + F - E = 2 - 2g \), where \( g \) is the genus of the shape represented by the mesh.

(a) [3 marks] Prove that it is not possible for all the vertices of a closed genus-0 triangle mesh to have degree 6. What value of \( g \), the genus, would make this possible?

(b) [2 marks] Show how you can remove ALL valence-3 and valence-4 vertices from a closed triangle mesh, without changing the mesh geometry.

[Solution]

(a) Suppose it is possible, then we have \( 3F = 2E \) and \( 6V = 2E \). It implies that \( V + F - E = 0 \), which is a contradiction to \( V + F - E = 2 - 0 = 2 \). For genus \( g = 1 \), it is possible, as I explained in class. Take a regularly tessellated rectangle and then wrap it into a cylinder by making its top-bottom and left-right pairs of sides meet.

(b) See Fig. 3 from this paper: http://www.cs.sfu.ca/~haoz/pubs/aghdaii_cag12_567.pdf

[3 marks] Problem 2: Gauss-Bonnet Theorem for closed triangle meshes

Given a smooth surface, the Gaussian curvature at a point is defined to be the product of the two principal curvatures at the point. One of the most celebrated results in differential geometry is the Gauss-Bonnet Theorem, which relates the total Gaussian curvature of a closed (boundary-less) manifold with its Euler Characteristic \( \chi \). The Euler characteristic of a closed shape \( S \) is a topological property and is defined as \( \chi(S) = 2 - 2g(S) \), where \( g(S) \) is the genus of the shape \( S \). Now we are ready to state the Gauss-Bonnet Theorem,

\[
\int_S \kappa \ dA = 2\pi \chi(S),
\]

where the left-hand side denotes the total Gaussian curvature over the closed manifold \( S \).

Note that this result is somewhat surprising as it magically relates geometric properties, the left-hand side, of a manifold with a property, which is totally topological.

Given a vertex \( v \) in a triangle mesh, the discrete Gaussian curvature at \( v \) is defined to be

\[ \kappa(v) = 2\pi - \text{(sum of face angles at } v), \]

as shown in the figure to the right. Note that the Gaussian curvature at a vertex can be negative. Naturally, the Gaussian curvature at a point along an edge or in the interior of a face over the surface of a triangle mesh is zero. Does the Gauss-Bonnet Theorem hold in the discrete case for a closed triangle mesh with any genus? If so, prove your claim. Otherwise, provide a simple counter example.

[Solution] Note that the Gaussian curvature at a point on a mesh edge or in the interior of a mesh face is zero. Thus the integral Gaussian curvature is the sum of Gaussian curvatures at the mesh vertices. This is given by \( 2V\pi - F\pi = 2\pi (V - F/2) \). For a closed
triangle mesh, we have $2E = 3F$, thus by the Euler formula, we have $V + F - E = V - F/2 = \chi$. Hence the sum of Gaussian curvature is nothing but $2\pi\chi$.

[2 marks] Problem 3: Non-obtuse triangulation

A triangulation is non-obtuse if each triangle is non-obtuse, i.e., it contains no obtuse angles. For a set of $n > 3$ points in the plane, is it always possible to find a non-obtuse triangulation of the convex hull of the points set with the $n$ points as vertices?

[Solution] No. Take $n = 4$ and look at the set of four points on the right. One cannot come up with a non-obtuse triangulation.

[3 marks] Problem 4: Non-obtuse mesh

A triangle mesh is said to be non-obtuse if each triangle is non-obtuse. Describe how you can modify the Marching Cubes algorithm to produce a non-obtuse mesh. It is sufficient for your idea to be only a rough one without all the technical details.

[Solution] The key idea is to judiciously place the vertices along the cube edges or in the interior of the cubes to ensure non-obtusity. For example, choosing to always place an edge vertex at the midpoint of an edge will be helpful. As well, when a vertex is to be placed inside a cube, always placing it at the center helps. Of course, to enumerate and verify all cases takes effort. But as long as you come up with the above rough idea, you will receive full credit for this problem.

For more technical details, check out this paper:

https://www.cs.sfu.ca/~haoz/pubs/li_tr06.pdf

and focus on Figures 3 and 4.