Curvature on Triangle Meshes

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Overview

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• Quadratic Fitting Method
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Introduction

• Many computer graphics and vision applications require first/second order differential properties such as curvatures to be as accurate as possible
• Examples include scene segmentation, anisotropic remeshing, surface smoothing and object recognition
Introduction

• Anisotropic remeshing of hand model
• Ridge lines in David model
Curvature Definitions

• Let $S$ be a surface in $\mathbb{R}^3$ described by an arbitrary parameterization of 2 variables.
• Define local tangent plane at a point $P$ orthogonal to the normal vector $N$.
• Curvatures, local bending of the surface.
• Normal curvature $\kappa N(\theta)$ for each unit direction $e\theta$. 
Curvature Definitions

- Principal curvatures $\kappa_1, \kappa_2$ associated with the orthogonal directions $e_1$ and $e_2$: the extreme values of normal curvatures.
- Mean Curvature $K_H$ is defined as the average of normal curvatures around point $P$.
- Gaussian Curvature $K_G$ is defined as the product of $\kappa_1$ and $\kappa_2$. 
Curvature Definitions
Quadratic Fitting Method

- Fit a surface to local points around the vertex in question using a quadratic polynomial surface.
- Curvature measurements of the fitted surface are used as curvature estimates for the point.
Quadratic Fitting Method

1. Estimate the normal $N$ at the point $P$ in question.
2. Create a rotation matrix that will map the global coordinate system to a local coordinate system at $P$ where the $Z$ axis of this new LCS will be aligned with the normal $N$ and the $X$ axis with the global $X$ axis. The rotation matrix $R = [r1, r2, r3]$ is defined as follows:

   \[ r3 = n, \quad r1 = \frac{(I - nn^T)i}{\| (I - nn^T)i \|}, \quad r2 = r3 \times r1, \]

   where $I$ is the identity matrix and $i$ is the global $X$ axis $[1,0,0]^T$.

3. Select the neighboring points to be used for surface fitting. The 1-Ring points around $P$ is a good choice.
4. Map the selected points from the GCS to the LCS using $R$: $x' = R(x - P)$
5. Solve for the coefficients by computing a least squared solution for:

   \[
   \begin{pmatrix}
   x_1^2 & y_1^2 & x_1y_1 \\
   \vdots & \vdots & \vdots \\
   x_n^2 & y_n^2 & x_ny_n
   \end{pmatrix}
   \begin{pmatrix}
   a' \\
   b' \\
   c'
   \end{pmatrix}
   =
   \begin{pmatrix}
   z_1 \\
   \vdots \\
   z_n
   \end{pmatrix}
   \]

   Since this is an overdetermined system in the form $Ax = b$ it can be solved by $x = (A^TA)^{-1}A^Tb$. 
Quadratic Fitting Method

6. We can now estimate the mean $KH$, Gaussian $KG$, and principal curvatures $K_1$ and $K_2$ with the following equations:

- $K_1 = a + c + ((a - c)^2 + b^2)^{1/2}$
- $K_2 = a + c - ((a - c)^2 + b^2)^{1/2}$
- $K_G = 4ac - b^2$
- $K_H = a + c$
Quadratic Fitting Method

Basic/Simple Fitting:
\[ Z = aX^2 + bXY + cY^2 \]

Extended Fitting:
\[ Z = aX^2 + bXY + cY^2 + dX + eY \]

Full Fitting:
\[ Z = aX^2 + bXY + cY^2 + dX + eY + f \]
(non-zero constant)
Quadratic Fitting Method

- With extended version, surface normal is refined to find better surface fit

- \( K_G = \frac{4ac - b^2}{(d^2 + e^2 + 1)^2} \)
- \( K_H = \frac{a+c+ae^2+cd^2 - bde}{(d^2 + e^2 + 1)^{3/2}} \)
Spatial Average Method

• Define properties of the surface at each vertex as *spatial averages* around this vertex.

• Restrict the average to be within the immediately neighboring triangles (the 1-ring).
Spatial Averages Method

- Area around vertex can be of 2 types
  - Voronoi
  - Barycenter
Spatial Averages Method

- Voronoi errors are preferred since error bounds are tight but only works with non-obtuse triangles
- We compromise by using a mixed area

\[ A_{\text{Mixed}} = 0 \]

For each triangle \( T \) from the 1-ring neighborhood of \( x \)

If \( T \) is non-obtuse, // Voronoi safe

// Add Voronoi formula (see Section 3.3)

\[ A_{\text{Mixed}^+} = \text{Voronoi region of } x \text{ in } T \]

Else // Voronoi inappropriate

// Add either area(\( T \))/4 or area(\( T \))/2

If the angle of \( T \) at \( x \) is obtuse

\[ A_{\text{Mixed}^+} = \text{area}(T)/2 \]

Else

\[ A_{\text{Mixed}^+} = \text{area}(T)/4 \]
Spatial Averages Method

- Integral of Mean Curvature
- Simplified using Gauss’ Theorem

\[ \iiint_{A_m} K(x) dA = \frac{1}{2} \sum_{j \in N_1(i)} (\cot \alpha_{ij} + \cot \beta_{ij}) (x_i - x_j), \]

- Average this using our \( A_{\text{mixed}} \)
Spatial Averages Method

Mean Curvature Normal Operator

\[ K(x_i) = \frac{1}{2A_{\text{Mixed}}} \sum_{j \in N_1(i)} (\cot \alpha_j + \cot \beta_j) (x_j - x_i) \]

The mean curvature is then half the magnitude of the vector.
Spatial Averages Method

- Using Gauss-Bonnet theorem:

\[
\int\int_{A_M} \kappa_G dA = 2\pi - \sum_{j=1}^{\#f} \theta_j
\]

- we then average this over our \( A_{\text{mixed}} \)
Spatial Averages Method

Gaussian Curvature Operator

\[ \kappa_G(x_i) = (2\pi - \sum_{j=1}^{#f} \theta_j) / A_{\text{Mixed}} \]
Spatial Averages Method

Principal Curvature Operators

\[
\begin{align*}
\kappa_1(x_i) &= \kappa_H(x_i) + \sqrt{\Delta(x_i)} \\
\kappa_2(x_i) &= \kappa_H(x_i) - \sqrt{\Delta(x_i)}
\end{align*}
\]

with: \( \Delta(x_i) = \kappa_H^2(x_i) - \kappa_G(x_i) \) and \( \kappa_H(x_i) = \frac{1}{2}\|K(x_i)\| \).
Spatial Averages Method

• Principal Directions is found by computing eigenvectors of curvature tensor \( B \):

\[
d_{i,j}^T B d_{i,j} = \kappa_{i,j}^N,
\]

\[
B = \begin{pmatrix}
a & b \\
b & c
\end{pmatrix}
\]

where \( d_{ij} \) is the unit direction in the tangent plane of the edge \( x_i x_j \)
Questions? Comments?

Time for Demo