On the Relation between Reiter's Default Logic and its (major) Variants

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Abstract. Default logic is one of the best known and most studied of the approaches to nonmonotonic reasoning. Subsequently, several variants of default logic have been proposed to give systems with properties differing from the original. In this paper we show that these variants are in a sense superfluous, in that for any of these variants of default logic, we can exactly mimic the behaviour of a variant in standard default logic. We accomplish this by translating a default theory under a variant interpretation into a second default theory wherein the variant interpretation is respected.

1 Introduction

Default logic [17] is one of the best known approaches to nonmonotonic reasoning. In this approach, classical logic is augmented by *default rules* of the form $\frac{\alpha:\beta_1,\ldots,\beta_n}{\gamma}$. Such a rule is informally interpreted as "if α is true, and $\beta_1,\ldots,\beta_n$ are consistent with what is known, then conclude γ by default". The meaning of a rule then rests on notions of provability and consistency with respect to a given set of beliefs. A set of beliefs sanctioned by a set of default rules, with respect to an initial set of facts, is called an *extension* of this set of facts.

However, the very generality of default logic means that it lacks several important properties, including *existence of extensions* [17] and *cumulativity* [12]. In addition, differing intuitions concerning the role of default rules have led to differing opinions concerning other properties, including *semi-monotonicity* [17] and *commitment to assumptions* [16]. As a result, a number of modifications to the definition of a default extension have been proposed, resulting in a number of variants of default logic. Most notably these variants include *constrained default logic* [18, 3], *cumulative default logic* [1], *justified default logic* [11], and *rational default logic* [15]. In each of these variants, the definition of an extension is modified, and a system with properties differing from the original is obtained.

In this paper we show that these variants are in a sense superfluous, in that each variant can be expressed within the framework of (the original) default logic. To accomplish this, we make use of translations mapping a default theory under a "'variant" interpretation onto a second theory under the interpretation of the original approach, such that the respectively resulting extensions are in a one-to-one correspondence. In the case of variant default logics that use the language of classical logic, we extend the language with labelled formulas. In the case of an *assertional default logic*, such as cumulative default logic, the situation is more complex since cumulative default logic makes use of "assertions," which extend the language of classical logic. Here we appeal to a quotation operator in which we can effectively name formulas; we then make assertions concerning quoted formulas by means of introduced predicates.

Hence we provide a unification of default logics, in that, we show that the original formulation of default logic is expressive enough to subsume its variants. The reverse relation does not to hold for constrained, justified, or cumulative default logic, in that one cannot express default logic in terms of these variants. However, rational default logic can be embedded in Reiter default logic, and vice versa. The translations that we provide show, in a precise sense, how each variant relates to standard default logic. As well, the approach lends some insight into characteristics of standard default theories. For example, our translations implicitly provide specific characterisations of default theories that are guaranteed to have extensions or are guaranteed to be semi-monotonic. That is, since we map variant default logics into default logic, the theories in the image of the mapping are guaranteed to retain properties of the original variant.

2 Default Logic and its Variants

Default logic [17] augments classical logic by *default rules* of the form $\frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma}$, where the constituent elements are formulas of classical propositional or first-order logic. Defaults with unbound variables are taken to stand for all corresponding instances. For simplicity, we deal just with *singular* defaults for which $n = 1$. A singular rule is *normal* if β is equivalent to γ; it is *semi-normal* if β implies γ. As regards standard default logic, [9] shows that any default rule can be transformed into a set of semi-normal defaults; similarly in constrained and rational default logic multiple justifications can be replaced by their conjunction. Moreover the great majority of applications use only semi-normal defaults, so the above assumption is a reasonable restriction. We denote the *prerequisite* α of a default $\delta = \frac{\alpha : \beta}{\gamma}$ by *Prereq*(δ), its *justification* β by *Justif*(δ) and its *consequent* γ by *Conseq*(δ). Conversely, to ease notation, in Section 3 we rely on a function δ to obtain the default rule in which a given prerequisite, justification, or consequent occurs, respectively. That is, for instance, $\delta(\mathit{Prereq}(\delta)) = \delta$.Moreover, for simplifying the technical results, we presuppose without loss of generality that default rules have unique components. To avoid confusion we will use the term *default logic* to refer solely to Reiter's original system. Variants will be referred to as constrained (cumulative, justified, etc.) default logic. Similar considerations apply to the notions of *default extension*.

A set of default rules D and a set of formulas W form a *default theory* (D, W) that may induce 0, 1, or multiple *extensions* in the following way.

Definition 1 ([17]). *Let (D, W) be a default theory. For any set S of formulas, let $\Gamma(S)$ be the smallest set of formulas such that*

1. $W \subseteq \Gamma(S)$,
2. $\Gamma(S) = \mathit{Th}(\Gamma(S))$,

3. *if* $\frac{\alpha:\beta}{\gamma} \in D$ *and* $\alpha \in \Gamma(S)$ *and* $S \cup \{\beta\} \not\vdash \bot$ *then* $\gamma \in \Gamma(S)$.

A set of formulas E *is an extension of* (D, W) *iff* $\Gamma(E) = E$.

That is, E is a fixed point of Γ. Any such extension represents a possible set of beliefs about the world at hand. For illustration, consider the default theories

$$(D_1, W_1) = \left(\left\{\frac{:B}{C}, \frac{:\neg B}{D}\right\}, \emptyset\right) \; ; \tag{1}$$
$$(D_2, W_2) = \left(\left\{\frac{:B}{C}, \frac{:\neg C}{D}\right\}, \emptyset\right) \; . \tag{2}$$

While (D_1, W_1) admits one extension, $Th(\{C, D\})$, the only extension of (D_2, W_2) is $Th(\{C\})$. In the literature (D_1, W_1) is often used as an illustrative example for what is sometimes referred to as *commitment to assumption* [16] (or: *regularity* [4]); similarly (D_2, W_2) illustrates *semi-monotonicity* [17].

Łukaszewicz [11] modifies default logic by attaching constraints to extensions in order to strengthen the applicability condition of default rules. A *justified extension* (called a *modified extension* in [11]) is defined as follows.

Definition 2 ([11]). *Let* (D, W) *be a default theory. For any pair of sets of formulas* (S, T) *let* $\Gamma(S, T)$ *be the pair of smallest sets of formulas* S', T' *such that*

1. $W \subseteq S'$,
2. $Th(S') = S'$,
3. *for any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $S \cup \{\gamma\} \cup \{\eta\} \not\vdash \bot$ *for every* $\eta \in T \cup \{\beta\}$ *then* $\gamma \in S'$ *and* $\beta \in T'$.

A set of formulas E *is a justified extension of* (D, W) *for a set of formulas* J *iff* $\Gamma(E, J) = (E, J)$.

So a default rule $\frac{\alpha:\beta}{\gamma}$ applies if all justifications of other applying default rules are consistent with the considered extension E and γ, and if additionally γ and β are consistent with E. The set of justifications J need not be deductively closed nor consistent.

In our examples, (D_1, W_1) has one justified extension, containing C and D. However, theory (D_2, W_2) has two justified extensions, one with C and one containing D.

In [18, 3] *constrained default logic* is defined. The central idea is that the justifications and consequents of a default rule jointly provide a context or assumption set for default rule application. The definition of a *constrained extension* is as follows.

Definition 3 ([3]). *Let* (D, W) *be a default theory. For any set of formulas* T, *let* $\Gamma(T)$ *be the pair of smallest sets of formulas* (S', T') *such that*

1. $W \subseteq S' \subseteq T'$,
2. $S' = Th(S')$ *and* $T' = Th(T')$,
3. *for any* $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $T \cup \{\beta\} \cup \{\gamma\} \not\vdash \bot$ *then* $\gamma \in S'$ *and* $\beta \wedge \gamma \in T'$.

A pair of sets of formulas (E, C) *is a constrained extension of* (D, W) *iff* $\Gamma(C) = (E, C)$.

Unlike Łukaszewicz's approach, the contextual information is here a deductively closed superset of the actual extension.

In our example, (D_1, W_1) has two constrained extensions, one containing C and another including D. Also, theory (D_2, W_2) has two constrained extensions, one with C and one with D.

The following is an alternative characterisation of *rational extensions*, originally proposed in [14], given in [10]:

Definition 4 ([14]). *Let (D, W) be a default theory. For any set of formulas T let $\Gamma(T)$ be the pair of smallest sets of formulas (S', T') such that*

1. $W \subseteq S' \subseteq T'$,
2. $S' = Th(S')$ *and* $T' = Th(T')$,
3. *for any* $\frac{\alpha : \beta}{\gamma} \in D$, *if* $\alpha \in S'$ *and* $T \cup \{\beta\} \not\vdash \bot$ *then* $\gamma \in S'$ *and* $\beta \wedge \gamma \in T'$.

A pair of sets of formulas (E, C) is a rational extension of (D, W) iff $\Gamma(C) = (E, C)$.

This definition is the same as that of constrained default logic, except for the consistency check. As with constrained default logic, (D_1, W_1) has two rational extensions, one containing C and one including D. However, theory (D_2, W_2) has only one rational extension with C.

Brewka [1] describes a variant of default logic where the applicability condition for default rules is strengthened, and the justification for adopting a default conclusion is made explicit. In order to keep track of implicit assumptions, Brewka introduces *assertions*, or formulas labeled with the set of justifications and consequents of the default rules which were used for deriving them. Intuitively, assertions represent formulas along with the reasons for believing them.

Definition 5 ([1]). *Let $\alpha, \gamma_1, \ldots, \gamma_m$ be formulas. An assertion ξ is any expression of the form $\langle \alpha, \{\gamma_1, \ldots, \gamma_m\} \rangle$, where $\alpha = Form(\xi)$ is called the asserted formula and the set $\{\gamma_1, \ldots, \gamma_m\} = Supp(\xi)$ is called the support of α.*[1]

To correctly propagate the supports, the classical inference relation is extended as follows.

Definition 6 ([1]). *Let $\mathcal{S}$ be a set of assertions. Then $\widehat{Th}(\mathcal{S})$, the assertional consequence closure operator, is the smallest set of assertions such that*

1. $\mathcal{S} \subseteq \widehat{Th}(\mathcal{S})$,
2. *if* $\xi_1, \ldots, \xi_n \in \widehat{Th}(\mathcal{S})$ *and* $Form(\xi_1), \ldots, Form(\xi_n) \vdash \gamma$ *then* $\langle \gamma, Supp(\xi_1) \cup \cdots \cup Supp(\xi_n) \rangle \in \widehat{Th}(\mathcal{S})$.

An *assertional default theory* is a pair $(D, \mathcal{W})$, where D is a set of default rules and $\mathcal{W}$ is a set of assertions. An *assertional extension* is defined as follows.

Definition 7 ([1]). *Let $(D, \mathcal{W})$ be an assertional default theory. For any set of assertions $\mathcal{S}$ let $\Gamma(\mathcal{S})$ be the smallest set of assertions $\mathcal{S}'$ such that*

[1] The two projections extend to sets of assertions in the obvious way. We sometimes misuse *Supp* for denoting the support of an asserted formula, e.g. $\langle \alpha, Supp(\alpha) \rangle$.

1. $\mathcal{W} \subseteq \mathcal{S}'$,
2. $\widehat{Th}(\mathcal{S}') = \mathcal{S}'$,
3. for any $\frac{\alpha:\beta}{\gamma} \in D$, *if* $\langle \alpha, Supp(\alpha) \rangle \in \mathcal{S}'$ *and* $Form(\mathcal{S}) \cup Supp(\mathcal{S}) \cup \{\beta\} \cup \{\gamma\} \nvdash \bot$
then $\langle \gamma, Supp(\alpha) \cup \{\beta\} \cup \{\gamma\} \rangle \in \mathcal{S}'$.

A set of assertions $\mathcal{E}$ *is an assertional extension of* $(D, \mathcal{W})$ *iff* $\Gamma(\mathcal{E}) = \mathcal{E}$.

For illustration, consider the assertional default theory (often used for illustrating the failure of *cumulativity* [12])

$$(D_3, W_3) = \left(\left\{ \frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A} \right\}, \emptyset \right). \tag{3}$$

This theory has one assertional extension, including $\langle A, \{A\} \rangle$ as well as $\langle A \vee B, \{A\} \rangle$. Adding the latter assertion to the set of assertional facts yields the assertional default theory

$$(D_4, W_4) = \left(\left\{ \frac{:A}{A}, \frac{A \vee B : \neg A}{\neg A} \right\}, \{ \langle A \vee B, \{A\} \rangle \} \right) \tag{4}$$

which has the same assertional extension. Note that without the support $\{A\}$ for $A \vee B$, one obtains a second assertional extension with $\langle \neg A, \{\neg A\} \rangle$. This is what happens in the previously-described default logics.

It is well-known that cumulative and constrained extensions are equivalent whenever the underlying facts contain no support. Similar relationships are given among original and Q-default logic [5], justified and affirmative [10], rational and CA-default logic [5], respectively (cf. [10]).

3 Correspondence with Constrained, Justified, and Rational Default Logic

This section presents encodings for representing major variant default logics in Reiter's default logic. For a default theory Δ, we produce a translated theory $\mathcal{T}_x \Delta$, such that there is a 1–1 correspondence between the extensions of Δ in x-default logic and (standard) extensions of $\mathcal{T}_x \Delta$. We begin with constrained and rational default logic, whose encoding is less involved, then consider that of justified default logic.

3.1 Correspondence with Constrained Default Logic

For a language $\mathcal{L}$ over alphabet $\mathcal{P}$, let $\mathcal{L}'$ be the language over $\mathcal{P}' = \{p' \mid p \in \mathcal{P}\}$. For a formula α, let α' be the formula obtained by replacing any symbol $p \in \mathcal{P}$ by p'; in addition define for a set W of formulas, $W' = \{\alpha' \mid \alpha \in W\}$.

Definition 8. *For default theory* (D, W)*, define* $\mathcal{T}_c(D, W) = (D_c, W_c)$ *where*

$$W_c = W \cup W' \qquad \textit{and} \qquad D_c = \left\{ \frac{\alpha : \beta' \wedge \gamma'}{\gamma \wedge (\beta' \wedge \gamma')} \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D \right\}.$$

Informally, we retain the justification of an applied default rule in an extension, but as a primed formula; this set of primed formulas then corresponds to the set C in Definition 3. Thus we essentially encode Definition 3, but in a standard default theory. Other variants of default logic are similarly encoded, although sometimes in a somewhat more complex formulation. For our examples in (1) and (2), we obtain:

$$\mathcal{T}_c(D_1, W_1) = \left(\left\{ \frac{: B' \wedge C'}{C \wedge B' \wedge C'}, \frac{: \neg B' \wedge D'}{D \wedge \neg B' \wedge D'} \right\}, \emptyset \right)$$
$$\mathcal{T}_c(D_2, W_2) = \left(\left\{ \frac{: B' \wedge C'}{C \wedge B' \wedge C'}, \frac{: \neg C' \wedge D'}{D \wedge \neg C' \wedge D'} \right\}, \emptyset \right) .$$

Now, theory $\mathcal{T}_c(D_1, W_1)$ yields two extensions in standard default logic, one containing $C \wedge B' \wedge C'$ and the other including $D \wedge \neg B' \wedge D'$. Analogously, we obtain two extensions from $\mathcal{T}_c(D_2, W_2)$, one with $C \wedge B' \wedge C'$ and the other with $D \wedge \neg C' \wedge D'$. In general, we have the following result.

Theorem 1. *For a default theory (D, W), we have that*

1. *if (E, C) is a constrained extension of (D, W) then $Th(E \cup C')$ is an extension of $\mathcal{T}_c(D, W)$;*
2. *if F is an extension of $\mathcal{T}_c(D, W)$ then $(F \cap \mathcal{L}, \{\varphi \mid \varphi' \in F \cap \mathcal{L}'\})$ is a constrained extension of (D, W).*

Theorem 2. *The constrained extensions of a default theory (D, W) and the extensions of the translation $\mathcal{T}_c(D, W)$ are in a 1–1 correspondence.*

3.2 Correspondence with Rational Default Logic

As expected, the mapping of rational default logic into standard default logic is close to that of constrained default logic:

Definition 9. *For default theory (D, W), define $\mathcal{T}_r(D, W) = (D_r, W_r)$ where*

$$W_r = W \cup W' \qquad \textit{and} \qquad D_r = \left\{ \frac{\alpha : \beta'}{\gamma \wedge (\beta' \wedge \gamma')} \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D \right\} .$$

As before, the consequent of rules in D_r encodes the formulas in a rational extension (Definition 4). For our examples in (1) and (2), we obtain:

$$\mathcal{T}_r(D_1, W_1) = \left(\left\{ \frac{: B'}{C \wedge B' \wedge C'}, \frac{: \neg B'}{D \wedge \neg B' \wedge D'} \right\}, \emptyset \right)$$
$$\mathcal{T}_r(D_2, W_2) = \left(\left\{ \frac{: B'}{C \wedge B' \wedge C'}, \frac{: \neg C'}{D \wedge \neg C' \wedge D'} \right\}, \emptyset \right) .$$

As with theory $\mathcal{T}_c(D_1, W_1)$, theory $\mathcal{T}_r(D_1, W_1)$ yields two extensions, one containing $C \wedge B' \wedge C'$ and $D \wedge \neg B' \wedge D'$, respectively. In contrast to $\mathcal{T}_c(D_2, W_2)$, however, we obtain one extension from $\mathcal{T}_r(D_2, W_2)$, containing $C \wedge B' \wedge C'$.

In general, we have the following result.

Theorem 3. *For a default theory (D, W), we have that*

1. *if (E,C) is a rational extension of (D,W) then $Th(E \cup C')$ is an extension of $\mathcal{T}_r(D,W)$;*
2. *if F is an extension of $\mathcal{T}_r(D,W)$ then $(F \cap \mathcal{L}, \{\varphi \mid \varphi' \in F \cap \mathcal{L}'\})$ is a rational extension of (D,W).*

As with Theorem 2, one can show that the extensions of a default theory (D,W) and the translation $\mathcal{T}_r(D,W)$ are in a 1–1 correspondence.

3.3 Correspondence with Justified Default Logic

Define for a language $\mathcal{L}$ over alphabet $\mathcal{P}$ and some set S, the family $(\mathcal{L}^s)_{s \in S}$ of languages over $\mathcal{P}^s = \{p^s \mid p \in \mathcal{P}\}$ for $s \in S$. For $\alpha \in \mathcal{L}$ and $s \in S$, let α^s be the formula obtained by replacing every symbol $p \in \mathcal{P}$ in α by p^s; in addition define for a set W of formulas, $W^s = \{\alpha^s \mid \alpha \in W\}$.

In what follows, we let the set of default rules D induce copies of the original language.

Definition 10. *For default theory (D,W), define $\mathcal{T}_j(D,W) = (D_j, W_j)$ where*

$$W_j = W \cup \bigcup_{\zeta \in D} W^\zeta \textit{ and } D_j = \left\{ \frac{\alpha : (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)}{\gamma \wedge (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)} \;\middle|\; \delta = \frac{\alpha : \beta}{\gamma} \in D \right\} .$$

For simplicity, we write $\beta = \mathit{Justif}^\circ(\delta)$ whenever $\mathit{Justif}(\delta) = (\beta^\delta \wedge \gamma^\delta) \wedge (\bigwedge_{\zeta \in D} \gamma^\zeta)$.

Abbreviating the two default rules in both examples, (1) and (2), by $\delta 1, \delta 2$ and $\delta 1, \delta 4$, respectively, we get (after removing duplicates):

$$\mathcal{T}_j(D_1, W_1) = \left(\left\{ \frac{: B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 2}}{C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 2}}, \frac{: \neg B^{\delta 2} \wedge D^{\delta 2} \wedge D^{\delta 1}}{D \wedge \neg B^{\delta 2} \wedge D^{\delta 2} \wedge D^{\delta 1}} \right\}, \emptyset \right)$$
$$\mathcal{T}_j(D_2, W_2) = \left(\left\{ \frac{: B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 4}}{C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 4}}, \frac{: \neg C^{\delta 4} \wedge D^{\delta 4} \wedge D^{\delta 1}}{D \wedge \neg C^{\delta 4} \wedge D^{\delta 4} \wedge D^{\delta 1}} \right\}, \emptyset \right)$$

In standard default logic, theory $\mathcal{T}_j(D_1, W_1)$ results in one extension containing C, D, $B^{\delta 1}, C^{\delta 1}, D^{\delta 1}$, and $\neg B^{\delta 2}, C^{\delta 2}, D^{\delta 2}$. Unlike this, $\mathcal{T}_j(D_2, W_2)$ gives two extensions, one with $C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 4}$ and another including $D \wedge \neg C^{\delta 4} \wedge D^{\delta 4} \wedge D^{\delta 1}$.

We have the following general result.

Theorem 4. *For a default theory (D,W), we have that*

1. *if (E,J) is a justified extension of (D,W) then $F = Th\Big(E \cup \bigcup_{\zeta \in D} E^\zeta \cup \bigcup_{\beta \in J} \{\beta^{\delta(\beta)}\}\Big)$ is an extension of $\mathcal{T}_j(D,W)$;*
2. *if F is an extension of $\mathcal{T}_j(D,W)$ then $(F \cap \mathcal{L}, J)$ is a justified extension of (D,W), where $J = \{\beta \mid \beta = \mathit{Justif}^\circ(\delta) \text{ and } \delta \in GD(\mathcal{T}_j(D,W), F)\}$.*

$GD(\mathcal{T}_j(D,W), F)$ gives the set of default rules generating F; see the full version for a formal definition.

In analogy to Theorem 2, one can show that the extensions of a default theory (D,W) and the translation $\mathcal{T}_j(D,W)$ are in a 1–1 correspondence.

3.4 Correspondence with (Standard) Default Logic

We can show that there is a self-embedding for standard default logic to standard default logic, using the encoding of the previous subsection:

Definition 11. *For default theory* (D,W)*, define* $\mathcal{T}_d(D,W) = (D_d, W_d)$ *where*

$$W_d = W \cup \bigcup_{\zeta \in D} W^{\zeta} \text{ and } D_d = \left\{ \frac{\alpha : \beta^{\delta}}{\gamma \wedge (\beta^{\delta} \wedge \gamma^{\delta}) \wedge (\bigwedge_{\zeta \in D} \gamma^{\zeta})} \;\middle|\; \delta = \frac{\alpha : \beta}{\gamma} \in D \right\} .$$

One can show that this mapping results in extensions that are in a 1–1 correspondence to those of the original theory. That is, one obtains a result similar to that in Theorem 4. This embedding also illustrates in a different fashion how default logic and justified default logic relate. As well, this translation allows for embedding standard default logic into rational default logic, as made precise next.

Theorem 5. *For a default theory* (D,W)*, we have that*

1. *if* E *is an extension of* (D,W) *then* (F,F) *is a rational extension of* $\mathcal{T}_d(D,W)$*, where* $F = Th\Big(E \cup \bigcup_{\zeta \in D} E^{\zeta} \cup \bigcup_{\delta \in GD((D,W),E)} \{Justif(\delta)^{\delta}\}\Big)$*;*
2. *if* (F,F) *is a rational extension of* $\mathcal{T}_d(D,W)$ *then* $F \cap \mathcal{L}$ *is an extension of* (D,W)*.*

As before, one can show that the extensions of a default theory (D,W) and the translation $\mathcal{T}_d(D,W)$ are in a 1–1 correspondance.

For our examples in (1) and (2), we get:

$$\mathcal{T}_d(D_1,W_1) = \left(\left\{ \frac{: B^{\delta 1}}{C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 2}}, \frac{: \neg B^{\delta 2}}{D \wedge \neg B^{\delta 2} \wedge D^{\delta 2} \wedge D^{\delta 1}} \right\}, \emptyset\right)$$
$$\mathcal{T}_d(D_2,W_2) = \left(\left\{ \frac{: B^{\delta 1}}{C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 4}}, \frac{: \neg C^{\delta 4}}{D \wedge \neg C^{\delta 4} \wedge D^{\delta 4} \wedge D^{\delta 1}} \right\}, \emptyset\right) .$$

In contrast to the two rational extensions obtained from (D_1,W_1), theory $\mathcal{T}_d(D_1,W_1)$ results in one rational extension containing $C, D, B^{\delta 1}, C^{\delta 1}, D^{\delta 1}$, and $\neg B^{\delta 2}, C^{\delta 2}, D^{\delta 2}$. As well, $\mathcal{T}_d(D_2,W_2)$ gives one rational extension with $C \wedge B^{\delta 1} \wedge C^{\delta 1} \wedge C^{\delta 4}$.

Note that a corresponding mapping into justified or constrained default logic is impossible; this is not a matter of the specific translation but rather a principal impossibility.

Theorem 6. *There is no mapping* $\mathcal{T}$ *such that for any default theory* (D,W)*, we have that the extensions of* (D,W) *are in a 1–1 correspondance with the constrained/justified extensions of* $\mathcal{T}(D,W)$*.*

To see this, consider theory $\left(\left\{\frac{:B}{\neg B}\right\}, \emptyset\right)$, having *no* extension. On the other hand, it is well known that every default theory has at least *one* justified and constrained extension [11, 3].

Finally, we note that a correspondence, as expressed in Theorem 5, can be established between justified and constrained extensions; we omit the details.

4 Correspondence with Cumulative Default Logic

This section presents an encoding for representing cumulative default logic and cumulative extensions in default logic. In order to be able to talk about an assertion $\langle \alpha, \{\beta_1, \dots, \beta_n\}\rangle$ within a (classical, logical) theory, an assertion is *reified* as an atomic formula $\langle \cdot, \cdot \rangle^{re}$, where each argument is a reified formula that does not contain an instance of $\langle \cdot, \cdot \rangle^{re}$. Thus $\langle \alpha, \{\beta_1, \dots, \beta_n\}\rangle$ is represented in the object language as $\langle \alpha, \beta_1 \wedge \dots \wedge \beta_n \rangle^{re}$. So that translated assertions have appropriate properties, we employ a set of formulas Ax_{re} axiomatising the reified formulas:

Definition 12. *Ax_{re} is the least set containing instances of the following schemata:*

1. *If* $\vdash \alpha$ *then* $\langle \alpha, \emptyset \rangle^{re} \in Ax_{re}$.
2. $(\beta_1 \equiv \beta_2) \supset (\langle \alpha, \beta_1 \rangle^{re} \equiv \langle \alpha, \beta_2 \rangle^{re})$.
3. $\langle \alpha, \gamma \rangle^{re} \wedge \langle \alpha \supset \beta, \psi \rangle^{re} \supset \langle \beta, \psi \wedge \gamma \rangle^{re}$.

We have the following analogue of Definition 6:

Theorem 7. *If* $\langle \alpha_1, \beta_1 \rangle^{re}, \langle \alpha_2, \beta_2 \rangle^{re} \in \Gamma$ *and* $\{\alpha_1, \alpha_2\} \vdash \gamma$ *then* $\Gamma \vdash \langle \gamma, \beta_1 \wedge \beta_2 \rangle^{re}$.

From this we establish a correspondence between extensions of cumulative default logic and default logic. We first define correspondences between assertions and formulas of classical logic.

Definition 13.

For Γ a set of assertions, define $Re(\Gamma) = \{\langle \alpha, \beta \rangle^{re} \mid \langle \alpha, \beta \rangle \in \Gamma\}$.

For Γ a set of formulas of classical logic, define $Re^{-1}(\Gamma) = \{\langle \alpha, \beta \rangle \mid \langle \alpha, \beta \rangle^{re} \in \Gamma\}$.

For Γ a set of assertions, define $Re^{+}(\Gamma) = Re(\Gamma) \cup Form(\Gamma) \cup Supp(\Gamma) \cup Ax_{re}$.

Definition 14. *For assertional default theory $(D, \mathcal{W})$, define $\mathcal{T}_a(D, \mathcal{W}) = (D_a, W_a)$ where*

$$W_a = Re^{+}(\mathcal{W}) \qquad \textit{and} \qquad D_a = \left\{ \frac{\langle \alpha, \psi \rangle^{re} : \beta \wedge \gamma}{\langle \gamma, \psi \wedge \beta \wedge \gamma \rangle^{re} \wedge \beta \wedge \gamma} \;\middle|\; \frac{\alpha : \beta}{\gamma} \in D,\ \psi \in \mathcal{L} \right\}.$$

The superscript re on formulas or sets of formulas indicates that these (sets of) formulas are in the image of our mapping, and are intended to be components satisfying a definition of a (Reiter) default extension.

This translation nicely shows that the support of (reified) assertions is only needed for keeping track of underlying assumptions when adding default conclusions to the set of facts; the consistency check remains unaffected. In fact, the treatment of $\beta \wedge \gamma$ in Definition 14 is identical to that of $\beta' \wedge \gamma'$ in Definition 8.

Consider our examples in (1) and (2):

$$\mathcal{T}_a(D_3, W_3) = \left(\left\{ \frac{\langle \top, \psi \rangle^{re} : A}{\langle A, \psi \wedge A \rangle^{re} \wedge A}, \frac{\langle A \vee B, \psi \rangle^{re} : \neg A}{\langle \neg A, \psi \wedge \neg A \rangle^{re} \wedge \neg A} \;\middle|\; \psi \in \mathcal{L} \right\}, Ax_{re} \right)$$

$$\mathcal{T}_a(D_4, W_4) = \left(\left\{ \frac{\langle \top, \psi \rangle^{re} : A}{\langle A, \psi \wedge A \rangle^{re} \wedge A}, \frac{\langle A \vee B, \psi \rangle^{re} : \neg A}{\langle \neg A, \psi \wedge \neg A \rangle^{re} \wedge \neg A} \;\middle|\; \psi \in \mathcal{L} \right\}, Ax_{re} \cup \{\langle A \vee B, \{A\} \rangle^{re}\} \cup \{A \vee B\} \cup \{A\} \right)$$

Both theories $\mathcal{T}_a(D_3, W_3)$ and $\mathcal{T}_a(D_4, W_4)$ yield one extension in standard default logic, containing $\langle A, \{A\} \rangle^{re}$.

We have the following general result.

Theorem 8. *For an assertional default theory $(D, \mathcal{W})$, we have that*

1. *if $\mathcal{E}$ is an assertional extension of $(D, \mathcal{W})$, then $Th\big(Re^+(\mathcal{E})\big)$ is an extension of $\mathcal{T}_a(D, \mathcal{W})$;*
2. *if E is an extension of $\mathcal{T}_a(D, \mathcal{W})$, then $Re^{-1}(E)$ is an assertional extension of $(D, \mathcal{W})$.*

Similar to the previous results, we also have a 1-1 correspondence between the extensions of a default theory and the extensions of the translation.

The translation given here for cumulative default logic is different from the previous translations, which clearly yielded a polynomial increase in size of the translated over the original theory. In the present case, Definition 14 gives an infinite number of defaults (due to the presence of ψ in the formula schemata). However, in practice we can nonetheless work with a translated theory that is only a polynomial increase over the original. First, for an assertional extension $\mathcal{E}$ and its translated counterpart $Th\big(Re^+(\mathcal{E})\big)$, we clearly have a 1-1 mapping between the respective sets of generating defaults. Second, any instantiation of ψ in Definition 14 (corresponding to the support of the prerequisite) can only draw upon elements of W or consequents of members of D; hence the size of any translated rule will be bounded by $|W| \times |D|$. As a result, an intelligent default prover can be restricted to a subset of the translated theory that is at worst a polynomial increase in size over the original.

5 Concluding remarks

We have shown how variants of default logic can be expressed in Reiter's original approach. Similarly, we have shown that rational default logic and default logic may be encoded, one into the other. This work then complements previous work in nonmonotonic reasoning which has shown links between (seeming) disparate approaches. Here we show links between (seemingly) disparate variants of default logic. As well, the translations clearly illustrate the relationships between alternative approaches to default logic. In fact, there is a division between default logic and rational default logic on the one hand, and the remaining variants on the other, manifesting itself through the property of semi-monotonicity. Although it has often been informally argued that the computational advantages[2] of semi-monotonicity are offset by a loss of representational power, this claim has up to now not been formally sustained. The results reported in [9] provide another indication of the relation between semi-monotonicity and expressiveness: normal default logic is a semi-monotonic fragment of Reiter's default logic and strictly less expressive than default logic. The same can be stated about cumulativity, as prerequisite-free, normal default logic (which corresponds to parallel circumscription) is strictly less expressive than normal default logic.

Our contributions can also be seen as a refinement of the investigations of complexity and/or expressiveness conducted in [7, 19, 13, 8, 6, 9]. From the perspective of complexity, there were of course hints that such mappings are possible. First, it is well-known that the reasoning problems of all considered variants are at the second level

[2] Semi-monotonicity allows for incremental constructions, also guaranteeing the existence of extensions.

of the polynomial hierarchy [7, 19]. The same is true for the "existence of extensions" problem in default logic and rational default logic, while it is trivial in justified and constrained default logic (and analogously for the respective assertional counterparts). In view of the same complexity of reasoning tasks, observe that our impossibility claim expressed in Theorem 6 is about the non-existence of corresponding sets of extensions. This does not exclude the possibility of an encoding of incoherent Reiter or rational default theories in a semi-monotonic variant that, for instance, indicates incoherence through a special-purpose symbol. However, there would be no 1–1 mapping here, since for any justified or constrained extension containing this special-purpose symbol, there would be no corresponding standard or rational extension.

The most closely related work to our own is that of Tomi Janhunen [9], who has investigated translations among specific subclasses of Reiter's default logic. For instance, he gives a translation mapping arbitrary default theories into semi-normal theories, showing that semi-normal default theories are as expressive as general ones. Other translation schemes can be found in [13], where among others the notion of semi-representability is introduced. This concept deals with the representation of default theories within restricted subclasses of default theories over an extended language. Although semi-representability adheres to a fixed interpretation of default logic, one can view our results as semi-representation results among different interpretations of default theories. As regards future research, it would be interesting to see whether the results presented here lead to new relationships in the hierarchy of non-monotonic logics established in [9]. Also, a more detailed analysis of time and space complexity is an issue of future research.

The present work may also, in fact, lend insight into computational characteristics of default logic. For example, our mappings provide specific syntactic characterisations of default theories that are guaranteed to have extensions. That is, for example, constrained default theories are guaranteed to have extensions; hence default theories appearing in the image of our mapping (Definition 8) are guaranteed to have extensions.

Apart from the theoretical insights, the great advantage of mappings such as we have given, is that it suffices to have one general implementation of default logic for capturing a whole variety of different approaches. In this respect, our results allow us to handle all sorts of default logics by standard default logic implementations, such as DeReS [2].

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