1. 1-to-k Matchings

We saw in class that a bipartite graph $G = (L \cup R, E)$ with the bipartition $L, R$ where $|L| = |R|$, has a perfect matching if and only if the following condition holds: for every subset $S \subseteq L$, the size of the neighborhood $|\Gamma(S)| \geq |S|$.

We shall generalize this as follows. Suppose $G = (L \cup R, E)$ is a bipartite graph where $|R| = k \cdot |L|$, for some integer constant $k > 0$. Call a matching between $L$ and $R$ 1-to-$k$ if every vertex $x \in L$ is matched with exactly $k$ vertices in $R$ so that no two vertices $x \neq y \in L$ have a common matched node from $R$.

(a) Prove that $G$ has a 1-to-$k$ matching if and only if, for every subset $S \subseteq L$, we have $|\Gamma(S)| \geq k \cdot |S|$.

(b) Give an efficient algorithm (based on Max Flow) for finding a 1-to-$k$ matching in a given input graph $G$, when it exists. (Your algorithm should also determine if such a matching exists.)

2. Network flows

Suppose you’re in charge of the Mars mission. There are possible experiments $E_1, \ldots, E_m$ to perform on Mars. Each experiment $E_j$ brings the profit of $p_j$ dollars (integer amount). These experiments depend on the set of instruments $I = \{I_1, \ldots, I_n\}$, where each experiment $E_j$ depends on the subset $R_j \subseteq I$ of instruments. Taking instrument $I_k$ to Mars costs $c_k$ dollars (integer amount).

You want to maximize your net revenue which is the total income from all experiments performed minus the total cost of all instruments taken to space.

Solve this problem using network flows. Consider the following network $G$. Its vertices are source $s$, vertices $I_1, \ldots, I_n$, vertices $E_1, \ldots, E_m$, and the sink $t$. The source $s$ has a directed edge to each vertex $I_k$ with capacity $c_k$. Each vertex $E_j$ is connected to the sink $t$ with a directed edge of capacity $p_j$. Finally, for each $1 \leq k \leq n$ and $1 \leq j \leq m$, vertex $I_k$ is connected with a directed edge to $E_j$ iff $I_k \in R_j$; the edges between $I_k$ and $E_j$ are of infinite capacity.

(a) Let $(A, B)$ be any st-cut of the network $G$ defined above such that the capacity $c(A, B)$ is finite (i.e., no infinite-capacity edge goes from $A$ to $B$). Show that if some $E_j \in B$, then $I_k \in B$ for every $I_k \in R_j$. 


(b) Show how to determine the maximum net revenue from the capacity of the minimum cut of $G$ and the given $p_j$ values.

(c) Give an efficient algorithm to determine which experiments to perform and which instruments to carry. What is the running time of your algorithm in terms of $n, m,$ and $r = \sum_{j=1}^{m} |R_j|$?

3. Problems 6, 12, and 19 from Chapter 7 of [KT].

4. **NP-completeness** Prove that each of the following problems is $NP$-complete.

   (a) Given an undirected graph on $n$ vertices (for an even number $n$), decide if the graph contains a clique of size $n/2$.

   (b) Given a propositional formula $\phi(x_1, \ldots, x_n)$, decide if $\phi$ has at least 2 satisfying assignments.