Search to Decision

SAT: Given $\varphi(x_1, \ldots, x_n)$, decide if $\varphi$ is satisfiable.

SAT-Search: Given $\varphi(x_1, \ldots, x_n)$, find a sat ass't, if exists.

Claim: $\text{SAT} \in \text{P} \Rightarrow \text{SAT-Search}$ is also in poly time.

Proof:

$\varphi(x_1, x_2, \ldots, x_n)$

$n$ steps

$\varphi(0, x_2, \ldots, x_n)$

$\varphi(1, x_2, \ldots, x_n)$

$n \times \text{SAT-time} \leq \text{poly}(n)$
**Application:** Universal SAT algo

**Setting:** Know $\text{SAT} \in \text{Time}(n^c)$, but do not know an actual algo. What can we do?

**Thm:** $\text{SAT} \in \text{Time}(n^c) \Rightarrow$

can find an explicit SAT algo with runtime $\sim n^{c+2}$

**Pf:** [inspired by Levin’s universal search algo]

Universal SAT algorithm
Given \( \Phi \) of size \( n \)

\[
\text{for } i = 1 \text{ to } n^{c+1}, \text{ simulate } TM M_i \text{ on } \Phi \text{ for } n^{c+1} \text{ steps;}
\]

\[
\text{if } M_i(\Phi) \text{ outputs a sat asst for } \Phi
\]

\[
\text{Accept ( & Halt)}
\]

\[
\text{end}\text{for}
\]

\[\text{Reject}\]

\underline{Correctness analysis:} Some \( M_d \) solves SAT-Search in time \( O(n^{c+1}) \) [Search-to-Decision]

For \( i = d \), \( M_d \) is simulated for \( \geq n^{c+1} \) steps (as long as \( n \geq d \))
Time analysis: \[ \sum_{i=1}^{n} c^{+1} \leq n^{c+2} \].

Suppose SAT ∈ Time \((n^2)\). Do we get a practical algo from the universal SAT algo above?

No! SAT algo is likely to be a very large program \(\Rightarrow\) decided by TM Md for LARGE d. Guaranteed to work only for
Guaranteed to work only for
\[ n > A ! \]
\[ |M_d| \approx 1,000 \]
\[ d \approx 2,100 \]

---

Polynomial Hierarchy

Thm [Fortnow]:

\[
\text{SAT} \land \text{TISP} \cup \bigcirc \bigcirc
\]

Simult. time
& space

Pl.: Uses two ideas:

1. NTIme Hierarchy
2. Poly Hierarchy \( \leq \) (Alternation)
PH = NP, \( \exists \bar{x} \psi(x_1, \ldots, x_n) \notin NP, \forall \bar{x} \not\exists \psi(x_1, \ldots, x_n) \)

\( \exists \bar{x} \forall \bar{y} \psi(x, y) : \Sigma^p_2 \)

\( \forall \bar{x} \exists \bar{y} \psi(x, y) : \Pi^p_2 \)

\( \exists \psi \in \Sigma^p_2 \)

\( \exists x \text{ Unique-SAT} = \)

\( \exists \psi \mid \psi \text{ has exactly 1 sat. ass'} \)

\( \exists x \forall y (\psi(x) = T \land y \neq x \Rightarrow \psi(y) = F) \)
Ex: \( \text{Min-Circuit} = \{ C \mid C \text{ is a cut s.t. no smaller eqn. cut for } C \text{ exists} \} \)

\[ A \text{ smaller cuts } C' \]

\[ \exists \text{ input } x \text{ s.t. } C(x) \neq C'(x) \]

---

\[ \exists K : E \subseteq F \]

\[ \forall E \subseteq F \]

Claim: \( \text{PH} \subseteq \text{PSPACE} \)
(NP \leq \text{PSPACE})

\text{Exercise}.