CMPT 710/407 - Complexity Theory: Problem Set 1

Due: September 27 (at the beginning of the class)

Reminder: Be careful when discussing the homework questions with others. Make sure you don't give away the solution since the purpose is for every student to try to figure it out. Also avoid the temptation of looking for a solution on the internet. You are always welcome to ask me questions about the homework. Please attempt all problems.

1. Describe a *one-tape* TM deciding the language $L = \{www \mid w \in \{0,1\}^*\}$ in polynomial time. Give a high-level description of your algorithm. You do not need to write down the transition function δ, but your description should be detailed enough so that one can implement a TM algorithm using your high-level description. (Note: You may use *one*-tape only.)

2. Show that $\mathsf{PSPACE} \neq \mathsf{Time}(n)$. (This is actually a surprising result since we do not know if either class properly contains the other one.) Hint: Assume equality. Think what "strange" consequences this has in terms of "time speed-up".

3. Show that the language $\{0,1\}$ is complete for P, under polytime reductions. Generalize this to argue that *every* non-trivial language in P is P-complete under polytime reductions. Here, a language L is called *non-trivial* if $L \neq \emptyset$ and $\bar{L} \neq \emptyset$, where $\bar{L}$ denotes the complement of L. (This means that when talking about P-completeness, we need to use reductions that are more efficient than polytime, e.g., logspace reductions.)

4. In this question, you're asked to fill in the steps of the proof of the following result about the language $PAL = \{w \in \{0,1\}^* \mid \ w \text{ is a palindrome}\}$.
   Theorem *Any one-tape TM deciding the language PAL requires time $\Omega(n^2)$.*

   For the proof, we need the notion of a *crossing sequence*. Given a computation of some one-tape TM M on an input w, let us number the tape cells as follows: the portion of the tape staring with the input w is numbered by $0, 1, 2, \ldots$; the portion of the tape to the left of the input w is numbered by negative integers $-1, -2, -3, \ldots$. For any integer i, we observe the behaviour of our TM as it crosses between the tape cells i and $i+1$, moving right or left across the cut. Let $S_w(i)$ denote the string that records the states of the TM and the directions of its tape head as the tape head moves between cells i and $i+1$. For example, the string $S_w(i)$ can look like $(q_1, R), (q_2, L), (q_1, R), (q_1, L)$; this means that the TM is in state q_1 as it crosses from cell i to $i+1$ for the first time, then the TM is in state q_2 as it crosses from $i+1$ to i next time, and so on.

   The string $S_w(i)$ is called a crossing sequence.

(a) Show that for any two strings $x = x_1x_2$ and $y = y_1y_2$ such that $|x_1| = |y_1| = i$ and $S_x(i-1) = S_y(i-1)$, if a TM M accepts x and y, then it also accepts the strings x_1y_2 and y_1x_2.

(b) Let M be any one-tape TM deciding PAL. Let x and y be any two distinct strings of length n each. Consider the inputs $X = x0^nx^R$ and $Y = y0^ny^R$, where 0^n means a string of n zeros, and x^R means the reverse of the string x. Show that for every $n < i < 2n$, it must be the case that $S_X(i) \neq S_Y(i)$.

(c) For the TM M as in the previous item, show that there exists a constant $\epsilon > 0$ such that the following holds: For every $n < i < 2n$, the number of strings of the form $X = x0^nx^R$ (for x of length n) that have $|S_X(i)| < \epsilon n$ is less than $2^{n/2}$.

(d) Using what you've shown in the previous item, argue that there will always exist a string $X = x0^nx^R$, for large enough n, such that $|S_X(i)| \geq \epsilon n$ for every $n < i < 2n$. Conclude that the TM M will take time $\Omega(n^2)$ on that string X.