

Lecture 10: Some Constructions of Expander Graphs

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# 1 Ramanujan Graphs (Lubotzky, Phillips & Sarnak, and Margulis)

## 1.1 Some Bounds on Spectral Expansion

**Theorem 1** (Alon, Boppana) For an infinite family of  $d$ -regular graphs  $G$ ,  $\lambda_2(G) \geq \frac{2\sqrt{d-1}}{d} - o(1)$ , where  $o(1) \rightarrow 0$  as  $|V| \rightarrow \infty$ .

**Theorem 2** (Friedman '02) For any constant  $d$  and  $\epsilon > 0$ , a random  $d$ -regular graph has  $\lambda_2(G) \leq \frac{2\sqrt{d-1}}{d} + \epsilon$ , with probability  $\geq 1 - n^{-\Omega(\sqrt{d})}$ .

Ramanujan  $d$ -regular graphs are those graphs  $G$  for which  $\lambda_2(G) \leq \frac{2\sqrt{d-1}}{d}$ . Explicit constructions of such graphs are known. In the next section we give one such construction due to Lubotzky, Phillips, and Sarnak.

## 1.2 Construction of Ramanujan Graphs

Let  $p$  and  $q$  be two prime numbers such that  $p \equiv 1 \pmod{4}$ ,  $q \equiv 1 \pmod{4}$  and  $q \geq 2\sqrt{p}$ . Let

$$S = \{(a_0, a_1, a_2, a_3) \mid a_i \in \mathbb{Z}_p, a_0 > 1, a_0 \text{ is odd,} \\ a_1, a_2, a_3 \text{ are even, and} \\ a_0^2 + a_1^2 + a_2^2 + a_3^2 = p\}$$

By Number Theory, it can be shown that  $|S| = p + 1$ . It can also be shown that there is an  $i$  such that  $i^2 \equiv -1 \pmod{q}$ . Then define for every  $\alpha = (a_0, a_1, a_2, a_3) \in S$ , the matrix  $\tilde{\alpha} \equiv \begin{bmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{bmatrix} \pmod{q}$ . Note that  $\det(\tilde{\alpha}) \equiv p \pmod{q}$ .

**Definition 3** The group  $PGL(2, \mathbb{Z}_q)$  is the quotient group of all  $2 \times 2$  invertible matrices over  $\mathbb{Z}_q$  modulo  $\left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \lambda \neq 0 \right\}$ .

**Definition 4** The group  $PSL(2, \mathbb{Z}_q)$  is the quotient group of all  $2 \times 2$  matrices over  $\mathbb{Z}_q$  with determinant 1, modulo  $\{I, -I\}$  where  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

Define  $G = (V, E)$ , where  $V = PGL(2, \mathbb{Z}_q)$ , and  $E = \{(A, B) \mid \exists \alpha \in S \text{ s.t. } B = A\tilde{\alpha}\}$ . The degree of  $G$  is  $p + 1$ . The number of vertices is  $\approx q^3$ . This graph is Ramanujan, i.e.,  $|\lambda_2| \leq \frac{2\sqrt{p}}{p+1}$ .

1. When  $p$  is a non-square modulo  $q$ , then the Ramanujan graph constructed above is bipartite.

2. When  $p$  is a square modulo  $q$ , then by letting  $V = PSL(2, \mathbb{Z}_q)$ , we obtain a non-bipartite Ramanujan graph.

The expansion for these Ramanujan graphs is  $\frac{d}{2}$ , where  $d$  is the degree of the graph. This is still far from non-constructive bounds of expansion, like  $d - 1.01$ .

**Open Problem 5** *Construct an explicit  $d$ -regular graph for some  $d$ , with expansion  $d - \theta(1)$ .*

It is known how to construct (unbalanced) bipartite expander graphs with expansion  $(1 - \epsilon)d$  for a given  $\epsilon > 0$  [Capalbo, Reingold, Vadhan, & Wigderson], but it is still open how to construct *non-bipartite* expanders with such an expansion factor.

## 2 Zig-Zag Expanders (Reingold, Vadhan, Wigderson '02)

### 2.1 Idea

Start with a “constant-size” expander  $H$ . By applying certain operations on the graph, turn  $H$  into an expander of “large” size. The operation will increase the number of vertices, but keep the degree constant, and keep the spectral expansion bounded away from 1.

Recall our notation:  $(n, d, \lambda)$ -graph is a  $d$ -regular multigraph on  $n$  vertices, with spectral expansion  $\lambda$ , i.e.,  $\max_{i>1} |\lambda_i| \leq \lambda$ .

We will use the following operations:

- Squaring:  $G \longrightarrow G^2$
- Tensor product:  $G_1, G_2 \longrightarrow G_1 \otimes G_2$
- Zig-zag product:  $G, H \longrightarrow G \mathbb{Z} H$

### 2.2 Squaring

**Definition 6** *For a multigraph  $G = (V, E)$ , define  $G^2 = (V, E')$  where  $E' = \{(x, y) | \exists z \in V, (x, z), (z, y) \in E\}$  (there is a path of length exactly 2 between  $x$  and  $y$  in  $G$ ).*

**Lemma 7** *Let  $G$  be an  $(n, d, \lambda)$ -graph. Then  $G^2$  is an  $(n, d^2, \lambda^2)$ -graph.*

Note that Squaring improves spectral gap, worsens degree and does not affect the size of the graph.

### 2.3 Zig-Zag Product

We will define zig-zag product later in the course.

**Lemma 8 (Zig-Zag Lemma)** *Let  $G$  be an  $(n_1, d_1, \lambda_1)$ -graph, and let  $H$  be an  $(d_1, d_2, \lambda_2)$ -graph. Then  $G \mathbb{Z} H$  is an  $(n_1 d_1, d_2^2, f(\lambda_1, \lambda_2))$ -graph, where  $f(\lambda_1, \lambda_2) \leq \lambda_1 + \lambda_2 + \lambda_2^2$ .*

**Note:** Think of  $H$  as a “small” graph and of  $G$  as a “large” graph. Then  $G \mathbb{Z} H$  is a “bigger” graph such that

1. degree of  $G \otimes H$  is a  $d_2^2$ , i.e., smaller than  $d_1$ .
2. If  $\lambda_1$  and  $\lambda_2$  are sufficiently small, then  $G \otimes H$  is an expander.

## 2.4 Zig-Zag Construction: First Attempt

Let  $H$  be a  $(d^4, d, \lambda_0)$ -graph for  $\lambda_0 \leq \frac{1}{5}$ . Define the sequence of graphs  $G_t$ , for  $t = 1, 2, \dots$  as follows:

$$G_1 = H^2,$$

$$G_{t+1} = G_t^2 \otimes H.$$

**Claim 9** Degree of  $G_t$  is  $d^2$ ; its number of vertices is  $d^{4t}$ , and its spectral expansion is  $\leq \frac{2}{5}$ .

**Proof of claim:** By induction.

Base Case: for  $t = 1$ , the proof is easy.

Inductive Step: suppose  $G_t$  is an  $(d^{4t}, d^2, \lambda)$ -graph, with  $\lambda \leq \frac{2}{5}$ . Then  $G_{t+1}$  is  $(d^{4t}d^4, d^2, f(\lambda^2, \lambda_0))$ -graph, where  $f(\lambda^2, \lambda_0) \leq \lambda^2 + \lambda_0 + \lambda_0^2 \leq \frac{4}{25} + \frac{1}{5} + \frac{1}{25} = \frac{2}{5}$ .  $\square$

How explicit is our construction? Let  $N_t$  denote the number of vertices in  $G_t$ . Recall that  $N_t = d^{4t}$ . Let  $\text{time}(G_t)$  be the number of operations to compute one neighbor of a given vertex in  $G_t$ . We have  $\text{time}(G_{t+1}) \approx 2\text{time}(G_t)$ . This implies that  $\text{time}(G_t) \approx 2^t = \text{poly}(N_t)$ .

We use the tensor product operation to make our expander construction more explicit, i.e, reducing  $\text{time}(G_t)$  from  $\text{poly}(N_t)$  to  $\text{polylog}(N_t)$ .

## 2.5 Tensor Product

**Definition 10 (Tensor Operation on Graphs)** For regular graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , define  $G_1 \otimes G_2 = (V, E)$ , where  $V = V_1 \times V_2$ ,  $E = \{((u_1, u_2), (v_1, v_2)) \mid (u_1, v_1) \in E_1, (u_2, v_2) \in E_2\}$ .

**Definition 11 (Tensor Operation on Vectors)** For every  $x = (x_i) \in \mathbb{R}^n$  and  $y = (y_i) \in \mathbb{R}^m$ , their tensor product is  $z = x \otimes y \in \mathbb{R}^{nm}$ , where  $z_{i,j} = x_i y_j$ . Equivalently, we can say if  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_m)$  then  $z = (x_1.y, x_2.y, \dots, x_n.y)$ .

**Definition 12 (Tensor Operation on Matrices)** For every matrix  $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$  and matrix  $B = (b_{i,j}) \in \mathbb{R}^{m \times m}$ , their tensor product is  $A \otimes B = (c_{i,j}) \in \mathbb{R}^{nm \times nm}$ , where  $c_{i'j', ij} = a_{i,j} b_{i',j'}$ . Equivalently, we can write

$$A \otimes B = \begin{pmatrix} a_{1,1}.B & a_{1,2}.B & \dots & a_{1,n}.B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}.B & a_{n,2}.B & \dots & a_{n,n}.B \end{pmatrix}.$$

**Exercises 13** (Properties of the tensor product)

1. For probability distributions  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$ ,  $x \otimes y$  is a probability distribution on  $[n] \times [m]$  (where  $[n] = \{1, 2, \dots, n\}$ ).
2. If  $A_1$  and  $A_2$  are normalized adjacency matrices of graphs  $G_1$  and  $G_2$  respectively, then  $A_1 \otimes A_2$  is the normalized adjacency matrix of  $G_1 \otimes G_2$ .
3. For every  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{m \times m}$ ,  $x \in \mathbb{R}^n$ , and  $y \in \mathbb{R}^m$ ,  $(A \otimes B)(x \otimes y) = (Ax) \otimes (By)$ .