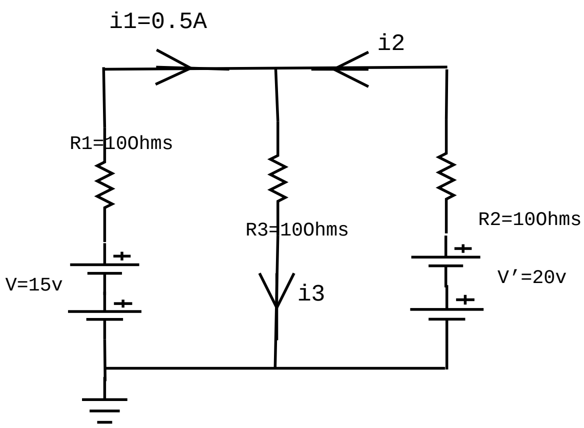
CMPT 881: Pseudorandomness Prof. Valentine Kabanets

Lecture 4: An application of Electrical Networks to USTCON

September 16, 2004 *Scribe: Ladan Mahabadi*

1 Electrical Networks



Kirchoff's Laws

[Current Law]: The sum of currents flowing into a junction equals the sum of currents flowing out of the junction (e.g $i_3 = i_1 + i_2$ in the above figure). In other words, no current gets lost, and injected current is equal to the collected current.

[Voltage Law]: The sum of voltages around a closed circuit is zero (e.g. $15 = v_1 + v_3$ where v_1 and v_3 are the voltage drops across R_1 and R_3 respectively).

Ohm's Law

$V = IR$ where V represents the voltage drop across resister R and I is the current passing through R. (e.g. voltage drop across $R_1 = v_1 = i_1 R_1 = 5$)
Given these two laws the other unknown values (e.g in the above figure) can be determined.

Example 1 $R_3 = 10$, $15 = v_1 + v_3 = 5 + v_3 \Rightarrow v_3 = 10$ and $i_3 = 1$ which then results in $i_2 = i_3$ - $i_1 = 0.5$

Thus, as demonstrated in the above example, one can determine unknown values such as the current values in terms of other values of the network such as the value of the resisters. The following demonstrates this fact for serial and parallel networks:

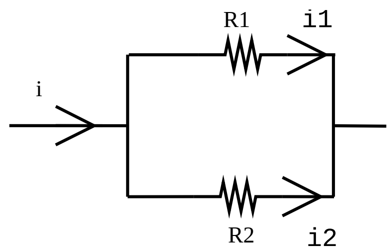


Figure 1: An example of a parallel circuit

Figure 2: An example of a serial circuit

Parallel Case (Figure 1):

- $V_1 = V_2 = V$

- $i_1 = \frac{R_2}{R_1+R_2} i$

  since in this parallel network $v_1 = v_2$ and $i_1 + i_2 =$ i, and by Ohm's law, $R_1 i_1 = R_2 i_2$. Therefore, $R_1 i_1 = R_2$(i - i_2); consequently, $(R_1 + R_2) i_1 = R_2 i$, and $i_1 = \frac{R_2}{R_1+R_2} i$.

- $i_2 = \frac{R_1}{R_1+R_2} i$

  Similar to the above calculation.

When $R_1 = R_2$, the current splits into two equal currents $i_1 = i_2 = \frac{i}{2}$

Serial Case (Figure 2):

- V $= V_1 + V_2$

- $V_1 = \frac{R_1}{R_1+R_2} V$

- $V_2 = \frac{R_2}{R_1+R_2} V$

  Where V_j represents the voltage drop across resister R_j.

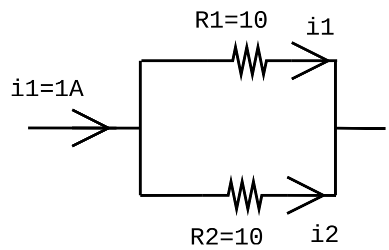


Figure 3: An example

From now on, we consider only circuits without any voltage sources.

Definition 2 *The* **effective resistance** *between two nodes u and v is the voltage difference between u and v when a current of 1 amp is injected into u and collected from v).*

As apparent in figure 3, i $= i_1 + i_2 = 2i_1 = 2i_2 = 1$ Amp $\Rightarrow i_1 = i_2 = \frac{1}{2}$
$V_1 = 10(0.5) = 5 = V_2 =$ V.
Effective resistance $= \frac{V}{i} = \frac{5}{1} = 5 < 10$. Thus, the effective resistance can be at most the bigger of the two branch resistances.

Application of Electrical Networks to Random Walks

After the above brief review of electrical networks, let's demonstrate the application of electrical networks to graphs via transforming connected graphs into electrical networks.

Counterparts of Graphs and Electrical Networks:

- A *connected graph* $\longrightarrow$ an *electrical network*.
- *Edges* $\longrightarrow$ 1 Ohm *resistors*
- *Hitting time* $\longrightarrow$ *Voltage*
- *Commute time* $\longrightarrow$ the *effective resistance*

Definition 3

- Hitting Time h_{uv}:

  The hitting time is the expected number of steps to reach v from u in $\geq$ 1 steps of a random walk. In other words, if a random variable S represents the number of steps required to reach v from u, then the Hitting time $= E[S]$

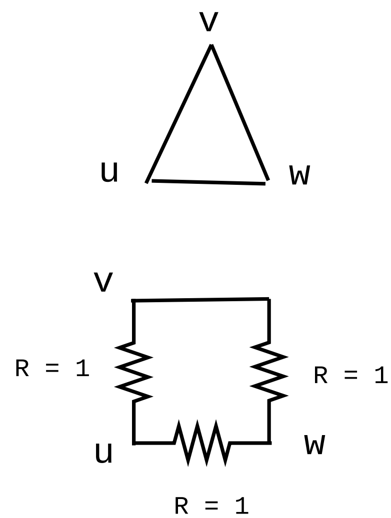


Figure 4: Transformation from a graph to an electrical network

- Commute Time C_{uv}:

  $C_{uv} = h_{uv} + h_{vu} = C_{vu}$ *(symmetrical)*
  In other words, the commute time is the expected number of steps from u back to u after at least *1 visit to v.*

- Cover$_u(G)$:

  The $Cover_u(G)$ *represents the expected number of steps from u that terminates upon visiting all vertices of G.*

- Cover(G):

  $Cover(G) = max_{u \in V}\{Cover_u(G)\}$

Notation

- d(x):
  For a node x, its degree, represented by d(x), is the number of x's neighbors in graph G.

- N(x):
  N(x) is the set of neighbors of vertex x in graph G.

In the following theorem, for convenience assume that graph G does not contain any self loops. However, note that a generalization to such graphs is possible.

Theorem 4 *Let G be an undirected, connected graph without self loops on n vertices, and m edges. Let $\widehat{G}$ be the electrical network corresponding to G. Then, for any two distinct vertices u and v of G, the following holds:*

- $h_{uv} = \varphi_{uv}$ *= the voltage between u and v when $d(x)$ Amps are injected into each node x of the electrical network and $2m$ Amps are collected from v*

- $C_{uv} = 2mR_{uv}$ *where R_{uv} is the effective resistance between u and v.*

Proof: Part 1:

- *In G:*
By definition of h_{uv}, $\forall u, v$ such that $u \neq v$,
h_{uv} = expected number of steps = $\frac{1}{d(u)}\Sigma_{N(u)}(1 + h_{wv})$. Note that for this equation to work, assume $h_{vv} = 0\ \forall v$. In general, $h_{vv} \neq 0$. So, this theorem will give correct values only for h_{uv} where $u \neq v$.

  Exercises 5 *Justify the above equation by arguing memorylessness (i.e lack of significance of the history) of a random walk.*

  Now, it follows that:
h_{uv} = expected number of steps = $\frac{1}{d(u)}\Sigma_{N(u)}(1 + h_{wv})$
d(u)h_{uv} = d(u) + $\Sigma_{w\in N(u)}h_{wv}$
Since $|N(u)|$ = d(u) by definition, it follows that for each $v \in$ V, and once v is fixed $\forall\ u \neq$ v:

$$\Sigma_{w\in N(u)}(h_{uv} - h_{wv}) = d(u) \tag{1}$$

- *In $\widehat{G}$*:
Inject i = d(x) Amps into each vertex x and collect from v.

  - *Kirchhoff's Law*:

    $V_{uw_i} = V_{uv} - V_{w_iv}$ where V_{ab} is defined as the voltage drop across the resistor on the edge connecting a to b
By **Ohm's Law**: $i_j = V_{uw_j}$ since $R_{uw_j} = 1$ by construction.

  - *Current conservation*:

$$d(u) = i_1 + i_2 + ... + i_{d(u)} = \Sigma_{w\in N(u)}(\varphi_{uv} - \varphi_{wv}) \tag{2}$$

$$\varphi_{vv} = 0\ \ \forall v \in V \tag{3}$$

Thus, equations (1) and (2) are identical once we identify φ_{uv} and h_{uv}. ■

Exercises 6 *Prove using linear algebra that the current system of equations has a unique solution.*

Proof: Part 2:
In this part, two circuits will be "super imposed" so that their voltages add up. Recall that $C_{uv} = h_{uv} + h_{vu}$ and by part 1, we have that:
$\mathrm{d}(u) = \Sigma_{w \in N(u)}(h_{uv} - h_{wv}) = \Sigma_{w \in N(u)}(\varphi_{uv} - \varphi_{wv})$
Thus, it must be that:

1. $h_{uv} = \varphi_{uv}$
   = voltage between u and v when a current of $\mathrm{d}(x)$ Amps is injected into each x and a current of $2m$ Amps removed from v.

2. $h_{vu} = \varphi_{vu}$
   = voltage between v and u when $\mathrm{d}(x)$ Amps are injected into each x and $2m$ Amps removed from u
   = voltage between u and v when $2m$ Amps are injected into u and $d(x)$ Amps are removed from each x.

By "superimposing" the two networks (so that the currents add up), we get a new electrical network where $2m$ Amps are injected into u and removed from v. So, we have $C_{uv} = \varphi_{uv} + \varphi_{vu} = 2mR_{uv}$ by Ohm's Law. ■

Note: The super imposing of the two networks is in such a way that the voltages get added up respectively, but the resistances remain the same.

Rayleigh's Short-cut principle:

- Reducing resistance on an edge (e.g "shorting" the two ends), can only decrease the effective resistance.

- Increasing resistance on an edge (e.g cutting the edge), can only increase the effective resistance.

For any distinct nodes u and v of a connected, undirected graph G, $R_{uv} \leq$ length of the shortest path from u to $v \leq n - 1$.
Now, since $C_{uv} = 2mR_{uv}$, it follows that $h_{uv} \leq h_{uv} + h_{vu} = C_{uv} \leq 2m(n-1)$ which for a *simple graph* with $|E|$ = m = $\frac{n(n-1)}{2}$ will be $\leq$ to $2(\frac{n(n-1)}{2}(n-1)) < n^3$.

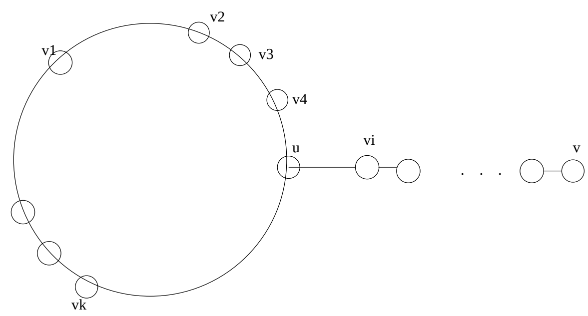


Figure 5: An example of a Lollipop Graph

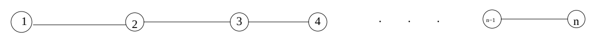


Figure 6: An example of a linear graph transformed into an electrical network where the vertices are labeled 1 through n and there is a resistance of 1 Ohm on each edge

Theorem 7 *For any simple, undirected graph G on n vertices, if there is a path from s to t in this graph then (via Markov's inequality):*
$\Pr_{randomwalkfromswillnotvisittin \leq 2n^3 steps} [\leq] \frac{1}{2}$

The above algorithm runs in RL (randomized $LOGSPACE$), and thus leads to the following corollary.

Corollary 8 $USTCON \in RL$

Example 9 *The Lollipop Graph:*
Figure 5 depicts a Lollipop graph that has a complete graph on $\frac{n}{2}$ vertices in the circular portion of the graph and $\frac{n}{2}$ more vertices on a line.It can be shown that:
$h_{vu} = \theta(n^2) \neq h_{uv} = \theta(n^3)$, and as a result, $C_{uv} = \theta(n^3)$.

Exercises 10 *Compute h_{vu} and h_{uv} in figure 5 using electrical networks*

Example 11 *Linear Graphs:*
In a linear graph, $C_{uv} = 2mR_v = 2(n-1)n = \theta(n^2)$ since for a traversal from u to v and back to u all of the edges must be traversed in a linear fashion one after the other.

Exercises 12 *Show that $h_{1n} = (n-1)^2$ in figure 6.*

As apparent from the above examples, adding edges may or may not reduce the commute time. However, adding loops will increase the cover time.
For a complete graph K_n, probability of getting to v from any vertex is $\frac{1}{n}$ due to the completeness of the graph.

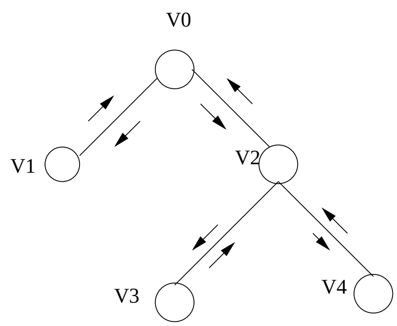


Figure 7: A spanning tree T

Exercises 13 *Show that for a complete graph K_n, $Cover = \theta(n \log n)$*
(Hint: Use the coupon collector's problem [1], [2]).

Theorem 14 *For any undirected connected graph G with n vertices and m edges,*
$Cover(G) \leq 2m(n-1)$

Proof: Take a graph G (connected), and compute any spanning tree of G (i.e. subset of edges of tree where all the vertices are touched). Figure 7 is an example of such a spanning tree.
Start at any vertex, say v_0. Traverse the tree in the depth-first order. Each edge is visited twice and since the tree contains all of the vertices, the above is a traversal of all vertices of the graph. Take traversal of T from v_0 to v_0 so that each edge of T is traversed exactly once in each direction.

Claim 15 *$Cover_{v_0} \leq \Sigma_{j=0}^{2n-3} h_{v_j,v_{j+1}} = \Sigma_{(u,w)\in E(T)} C_{uw}$ where v_j, v_{j+1} are adjacent vertices.*

Exercises 16 *Prove the above claim.*

Recall that $C_{uv} = 2mR_{uw}$, and $R_{uw} \leq 1$ since $(u,w) \in E$, then $C_{uw} \leq 2m$.
So, since any spanning tree on n vertices has $n-1$ edges, we get $\Sigma_{(u,v)\in E(T)} C_{uw} \leq 2m(n-1)$

■

Note that the calculation did not depend on v_0. Thus, it works even in the worst case.

References

[1] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein, **Introduction to Algorithms**, The MIT Press, 2nd edition, September 2001.

[2] Rajeev Motwani, Prabhakar Raghavan, **Randomized Algorithms**. Cambridge University Press, August 1995.