CMPT 881: Pseudorandomness	Prof. Valentine Kabanets
Lecture 7: Spectral Expansion I	
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1 Spectral Expansion

Let G be a d-regular multigraph on n vertices and A be the normalized adjacency matrix of G. Set $\lambda_2(G) = \max\{|\lambda_i| \; i \geq 2\}$ where $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are eigenvalues of G.

Definition 1 *G has spectral expansion λ, if $\lambda_2(G) \leq \lambda$.*

Theorem 2 *Spectral expansion $\Rightarrow$ Vertex expansion .*
If G has spectral expansion λ then $\forall \alpha < 1$, G has vertex expansion $(\alpha n, \frac{1}{(1-\alpha)\lambda^2+\alpha})$.

We require some definitions and lemmas to prove the Theorem.

Definition 3 *For any probability distribution π, the collision probability of π is defined as:*

$$Col(\pi) = \sum_{i=1}^{i=n} \pi_i^2.$$

Definition 4 *For any probability distribution π, $Supp(\pi) = \{i \mid \pi_i > 0\}$*

In what follows let $u = (\frac{1}{n}, \ldots, \frac{1}{n})$. u is uniform distribution.

Lemma 5 *For any distribution π*
1) $Col(\pi) \geq \frac{1}{|Supp(\pi)|}$.
2) $Col(\pi) = \|\pi - u\|^2 + \frac{1}{n} = \|\pi - u\|^2 + Col(u)$.

Remark 6 For distribution which is uniform over $Supp(\pi)$, the collision probability is

$$\sum_{i \; \in \text{Supp}(\pi)} \frac{1}{|Supp(\pi)^2|} = \frac{1}{|Supp(\pi)|}.$$

It means that among all probability distribution with same Support, the uniform distribution has minimum collision probability.

Proof of Lemma 5:

Proof of part (1): By Cauchy-Schwarz, $\forall$ vectors u, v, $(u, v) \leq \|u\|\|v\|$, where (u, v) is the inner product of u, v.

Define vector x as follow:

$$x_i = \begin{cases} 1 & \text{if } \pi_i > 0 \\ 0 & \text{if } \pi_i = 0 \end{cases}.$$

We have $\sum x_i^2 = |Supp(\pi)|$. Therefore $1 = (\pi, x)^2 \leq \|\pi\|^2 \|x\|^2 = Col(\pi)|Supp(\pi)|$.

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Proof of part (2): Write $\pi = u + \pi^\perp$. Then $Col(\pi) = \|\pi\|^2 = \|u + \pi^\perp\|^2 = (u + \pi^\perp, u + \pi^\perp) = (u, u) + 2(u, \pi^\perp) + (\pi^\perp, \pi^\perp) = \|u\|^2 + \|\pi^\perp\|^2 = \frac{1}{n} + \|\pi - u\|^2$.

Note that $(u, \pi^\perp) = 0$ from previous lecture! ■

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Proof of Theorem 2:

$\forall \pi$, $\|A\pi - u\|^2 \leq \lambda^2 \|\pi - u\|^2$. Let π be uniform distribution on set S of size $l \leq \alpha n$. By Lemma 5 $|N(S)| = |Supp(A\pi)| \geq \frac{1}{Col(A\pi)}$. So we need to upperbound $Col(A\pi)$.

$$Col(A\pi) = \|A\pi - u\|^2 + \frac{1}{n} \leq \lambda^2 \|\pi - u\|^2 + \frac{1}{n} = \lambda^2 (Col(\pi) - \frac{1}{n}) + \frac{1}{n}.$$

So we have

$$Col(A\pi) - \frac{1}{n} \leq \lambda^2 (Col(\pi) - \frac{1}{n}).$$

Finally,

$$\frac{1}{|N(S)|} - \frac{1}{n} \leq Col(A\pi) - \frac{1}{n} \leq \lambda^2 (\frac{1}{|S|} - \frac{1}{n}).$$

Therefore we have $\frac{|S|}{|N(S)|} \leq \lambda^2 + |S|\frac{1-\lambda^2}{n}$, and $\frac{|N(S)|}{|S|} \geq \frac{n}{\lambda^2 n + |S|(1-\lambda^2)} \geq \frac{n}{\lambda^2 n + \alpha n(1-\lambda^2)} = \frac{1}{\lambda^2 + \alpha(1-\lambda^2)}$.

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Theorem 7 *Vertex expansion ⇒ Spectral expansion [1]*
$\forall$ constant $\gamma > 0$, $d > 2$, there is $\lambda < 1$, s.t. if G is a d-regular graph with n vertices and $(\frac{n}{2}, 1 + \gamma)$-expansion, then it has $\lambda \leq 1 - \Omega(\frac{\gamma^2}{d^2})$ spectral expansion.

References

[1] N. Alon, Z. Galil and V. D. Milman, Better expanders and superconcentrators, J. Algorithms 8 (1987), 337-347.