1 Two Notions of Spectral Expansion

Let $G$ be a $d$-regular graph on $n$ vertices with normalized adjacency matrix $A$. Let $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ eigenvalues of $G$.

a. $\lambda_2(G) = \max_{i \geq 2} \{|\lambda_i|\}$

b. $\lambda_2(G) = \max_{i \geq 2} \{\lambda_i\}$

If $\lambda_2 \ll 1$ we have an expander. We prove this next.

Definition 1 Let $G = (V, E)$. For $B, C \subset V$, let $e(B, C)$ be the number of ordered pairs $(u, v)$ where $u \in B$ and $v \in C$, and $(u, v) \in E$. If $B, C$ are disjoint then $e(B, C)$ is the number of edges between $B$ and $C$.

Theorem 2 Let $G = (V, E)$ be a $d$-regular graph on $n$ vertices, with normalized adjacency matrix $A$. Let $\lambda$ be the second largest eigenvalue of $A$. Then for every partition of $V$ into two disjoint sets $B, C$ $\frac{e(B, C)}{nd} \geq (1 - \lambda)\mu(B)\mu(C)$, where $\mu(B) = \frac{|B|}{n}$ and $\mu(C) = \frac{|C|}{n}$.

Remark Here is an interpretation of Theorem 2. Consider the following two random experiments. Experiment 1: pick a random vertex $u \in V$ of the graph $G$, and then pick one of its $d$ neighbors $v$, uniformly at random. Experiment 2: pick a random vertex $u \in V$ and then pick a random vertex $v \in V$. What is the probability of picking an ordered pair $(u, v)$ such that $u \in B$ and $v \in C$? For Experiment 1, it is $\frac{e(B, C)}{nd}$; for experiment 2, it is $\mu(B)\mu(C)$. So, Theorem 2 says that, for an expander graph $G$, Experiment 1 results in a pair from $B \times C$ with a probability that is a constant fraction (depending on $\lambda$) of the probability of picking such a pair in Experiment 2. For expander graphs where we have a bound on $\lambda_2(G)$ rather than just $\lambda_2$, we can prove that the probability of getting a pair from $B \times C$ in Experiment 1 is close to that in Experiment 2 — see the Expander Mixing Lemma below (Lemma 5).

Claim 3 For any real vector $x = (x_1, x_2, \ldots, x_n)$ and any graph $G$ of order $n$ (without self-loops\(^1\)), $(I - A)x, x = \frac{1}{d} \sum_{(i,j) \in E} (x_i - x_j)^2$.

\(^1\)The result is true for arbitrary graph. However, for the sake of simplicity, we prove it only for the case of loopless graphs.
Proof:
\[(I - A)x, x) = (x, x) - (Ax, x) = \sum_{i=1}^{i=n} x_i^2 - \frac{2}{d} \sum_{(i,j) \in E} (x_i x_j) = \frac{1}{d} \sum_{(i,j) \in E} (x_i^2 + x_j^2) - \frac{2}{d} \sum_{(i,j) \in E} x_i x_j = \frac{1}{d} \left[ \sum_{(i,j) \in E} (x_i - x_j)^2 \right].\]

Proof of Theorem 2: Let \( n = |V|, b = |B|, c = |C| = n - b. \)
Define \( x = (x_1, x_2, \ldots, x_n) \) by
\[x_i = \begin{cases} -c & \text{if } i \in B \\ b & \text{if } i \in C. \end{cases}\]
Observe that the eigenvalues of the matrix \( I - A \) are \( 0 = 1 - \lambda_1, 1 - \lambda_2, \ldots, 1 - \lambda_n. \) So \( 1 - \lambda \) is the second smallest eigenvalue of \( I - A. \) By Rayleigh’s Theorem we have
\[1 - \lambda = \min_{y \perp u} \frac{(I - A)y, y}{\|y\|^2} \leq \frac{(I - A)x, x}{\|x\|^2},\]
where for the last inequality we used the fact that \( x \perp u \) by the definition of \( x. \) By Claim 3, we have \((I - A)x, x) = \frac{\sigma}{d} e(B, C). \) Hence
\[(I - A)x, x) \geq (1 - \lambda) \|x\|^2 = (1 - \lambda)bcn.\]
Therefore \( e(B, C) \geq (1 - \lambda) \frac{bcn}{n^2} \) i.e. \( \frac{e(B, C)}{nd} \geq (1 - \lambda) \mu(B) \mu(C).\]

Corollary 4 If \( \lambda \) is the second largest eigenvalue of a \( d \)-regular graph \( G \) on \( n \) vertices, then \( \forall S \subset V \) where \( |S| \leq \frac{n}{2}, \) we have \( |N(S) - S) | \geq \frac{(1 - \lambda) |S|}{2}. \)

Proof: By Theorem 2, \( e(S, \tilde{S}) \geq (1 - \lambda) |S||\tilde{S}| \right) \frac{d}{n} \geq (1 - \lambda) |S| \frac{d}{n} \tilde{S} = \frac{(1 - \lambda) |S|}{2}. \) Also number of vertices in \( \tilde{S} \) \( S \) that have neighbor in \( S \) is at least \( \frac{e(S, \tilde{S})}{d}. \)

By adding self-loops, we can get a \( (d+1) \)-regular graph such that \( \forall S, |S| \leq \frac{n}{2}, |N(S)| \geq (1 + \frac{1 - \lambda}{2}) \frac{|S|}{2}. \)

Lemma 5 (Expander-Mixing Lemma) Let \( G \) be a \( d \)-regular graph with \( \lambda_2(G) \leq \lambda. \) Then \( \forall S, T \subset V, |e(S, T) - \mu(S) \mu(T)| \leq \lambda \sqrt{\mu(S) \mu(T)}. \)

Proof: Let \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be the characteristic vector for \( S, \) where
\[\sigma_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}\]
Similarly, let \( \tau \) be the characteristic vector for \( T. \) Let \( A \) be the normalized adjacency matrix of \( G. \) Observe that \( e(S, T) = \sum_{\text{ordered pairs } (i,j) \in E} \sigma_i \tau_j = \sigma(dA) \tau. \) Let \( \alpha = \mu(S) \) and \( \beta = \mu(T). \)
Write \( \sigma = \sigma^\parallel + \sigma^\perp \), where \( \sigma \) is parallel to \( u \) and \( \sigma^\perp \) is orthogonal to \( u \). So \( \sigma = c u \) where 

\[
c = \frac{\langle \sigma, u \rangle}{\|u\|^2} = \frac{|S|}{\pi} = |S| = \alpha n.
\]

Similarly for \( \tau^\parallel = \beta n u \). Now,

\[
\frac{e(S, T)}{dn} = \frac{1}{n} [\sigma A \tau] = \frac{1}{n} (\langle \sigma^\parallel + \sigma^\perp \rangle A (\tau^\parallel + \tau^\perp)) = \frac{1}{n} [\sigma^\parallel A \tau^\parallel + \sigma^\parallel A \tau^\perp + \sigma^\perp A \tau^\parallel + \sigma^\perp A \tau^\perp] = \frac{1}{n} [\alpha n u A \beta n u + \sigma^\perp A \tau^\perp] = \alpha \beta + \frac{1}{n} \sigma^\perp A \tau^\perp.
\]

Note that \( \sigma^\perp A \tau^\parallel = \sigma^\parallel A \tau^\perp = 0 \). Therefore

\[
\left| \frac{e(S, T)}{dn} - \alpha \beta \right| = \frac{1}{n} ||\sigma^\perp|| A ||\tau^\perp|| \leq \frac{1}{n} ||\sigma^\perp|| ||\tau^\perp|| \leq \frac{\lambda}{n} ||\sigma|| ||\tau|| = \frac{\lambda}{n} \sqrt{\alpha \beta n \beta n} \leq \frac{\lambda}{n} \alpha \beta.
\]

So we have \( \left| \frac{e(S, T)}{dn} - \alpha \beta \right| \leq \lambda \sqrt{\alpha \beta} \).

**Remark** The Expander Mixing Lemma shows that a \( d \)-regular expander graph \( G = (V, E) \) on \( n \) vertices, with a bound on \( \lambda_2(G) \leq \lambda \), can be used to sample from the direct product \( V \times V \) in a way that is more randomness efficient than the obvious sampling from \( V \) two times independently. Namely, it says that, for any sets \( S, T \subseteq V \), if we take a random vertex \( u \in V \) and then take its random neighbor \( v \), we get \((u, v) \in S \times T \) with about the same probability as we would when sampling \( u \in V \) and \( v \in V \) independently. Thus, instead of spending \( 2 \log n \) random bits, we can spend only \( \log n + \log d \) random bits.

## 2 Random Walks on Expanders

Suppose \( A \) is a randomized algorithms with error probability \( \epsilon \) using \( m \) random bits. By running \( A \) \( t \) times, we can reduce the error exponentially, but we should use \( mt \) random bits. By walking randomly on an expander graph \( G \) with parameters \( (2^m, d, \lambda) \) (here \( 2^m \) is number of vertices, \( d \) is degree, \( \lambda \) is spectral expansion factor), we can reduce the number of random bits to \( m + O(t) \).

We start in a uniformly chosen vertex \( v \) and walk randomly in \( G \) for \( t \) steps, using the labels of the vertices encountered in the walk as a random strings for \( A \). Note that for finding a random neighbor we need to use \( \log d \) bits, which is a constant. The following table summarize the above observation.

<table>
<thead>
<tr>
<th>Random Algorithm</th>
<th>1-sided error probability</th>
<th>random bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( 1/10 )</td>
<td>( m )</td>
</tr>
<tr>
<td>Repeat ( A ), ( t ) times</td>
<td>( \leq 2^{-t} )</td>
<td>( tm )</td>
</tr>
<tr>
<td>Random walk on ( (2^m, d, 1/36) )</td>
<td>( \leq 2^{-t} )</td>
<td>( m + O(t) )</td>
</tr>
</tbody>
</table>