Abstract—We present a "separation" algorithm for achieving color constancy and theorems concerning its accuracy. The algorithm requires extra information from the optical system, over and above the usual three values mapping human cone responses. However, with this additional information—specifically, a sampling across the visible range of the reflected, color-signal spectrum impinging on the optical sensor—we are able to separate the illumination spectrum from the surface reflectance spectrum contained in the color-signal spectrum which is, of course, the product of these two spectra. At the heart of the separation algorithm is a general statistical method for finding the "best" illumination and reflectance spectra, within a space represented by finite-dimensional linear models of statistically typical spectra, whose product closely corresponds to the spectrum of the actual color signal. Using this method, we are able to increase the dimensionality of the finite-dimensional linear model for surfaces to a realistic value. One method of generating the spectral samples required for the separation algorithm is to use the chromatic aberration effects of a lens, and an example of this is given. We detail the accuracy achieved in a large range of tests and show that agreement with actual surface reflectance is excellent.

Index Terms—Color constancy, color perception, computer vision, finite-dimensional linear model, statistical methods.

I. INTRODUCTION

The term "color constancy" usually means the recovery of perceived surface color from the strengths of three receptor values representing cone responses of the visual system, RGB responses of a color camera or the like, independent of the color of the light illuminating the object. Being able to extract color descriptors that are independent of the illumination is desirable because of the variety of situations in which color is important, but illumination conditions cannot be controlled. Such situations include visually guided robots, automatic terrain classification by remote sensing, and perhaps somewhat less obvious, color reproduction [1], [2].

In color reproduction, the knowledge of surface spectral reflectance helps in setting the intensities of the three color guns in a video display so as to render the scene colors best. Images could even be shot under one ambient illumination and displayed as if they had been shot under a different illumination. Other uses for color constancy information include matching paints or dyes (especially at separate locations), helping a robot outdoors continue to see the grass as green even as a cloud appears and changes the illuminant spectrum, sorting ripe fruit from unripe, and estimating the ambient light spectrum for aid in correct filtering for photography—to name just a few applications.

The objective of any color constancy algorithm is substantial: one must effectively recover some representation of the entire surface reflectance spectrum at each pixel in order to make a firm statement on surface colors. Maloney and Wandell [3]–[5], [2], [1], [6] and Yuille [7] start with the idea of describing color spectra with finite-dimensional linear models [8]–[16]. The finite-dimensional linear-model approach condenses all spectral information into a few numbers by supposing that illumination and reflectance can each be approximated by weighted sums of basis functions. The basis functions for illumination model the principal characteristics of spectra of natural illuminants, such as daylight [17], while those for reflectance model the characteristics of a large set of spectral reflectances [18], [19].

Making use of the relationship between receptor values and basis-function weights, Maloney, Wandell, and Yuille show how to recover both the weights representing surface reflectance and those representing illumination. The method requires solving a set of equations based on the receptor values from several regions of different color. The method’s primary limitation is that a system of only three receptor classes is necessarily restricted to being able to model surfaces with only a two-dimensional vector space. But this dimensionality is too small to capture the richness of surface color.

In this paper, our goal is perhaps more modest in that in order to carry out our "separation" algorithm, we demand additional measurements be provided as input, over and above the response values of three photoreceptor classes. In particular, our algorithm requires a sampling of the entire (visible) color-signal spectrum. The product of the illumination spectrum with that of the surface reflectance comprises the color signal, the light reflected from the surface and entering the camera or eye.

From this additional information, though, our separation algorithm extracts the illumination and reflectance with higher dimensionality. It also applies to situations...
for which previous algorithms, based on receptor values alone, are incapable of supplying answers.

Given the sampled color signal, the separation algorithm divides it into one component due to the illumination, and a second due to the reflectance. In the result, the dimensionality of the vector space describing surfaces is increased to 3, while the same dimension of 3 for the illumination space and the number of receptor classes is maintained. Therefore, the separation algorithm can better model surface reflectances than previous models. In fact, provided that the basis functions for the surface and illumination vector spaces together form a set satisfying certain requirements given below, our analysis is unrestricted with regard to the dimensionality of each space. As well, we do not require the illumination to be spatially constant.

While it is straightforward to imagine how one might acquire the color-signal spectrum in machine applications, such as satellite imaging or color xerography, it seems on first consideration that such extra information is inaccessible in systems such as the human visual system. This, however, turns out not to be the case. In [20]-[22], it was shown that, in fact, spectral information can be gleaned from the chromatic aberration inherent in lens systems, including human ones. As an example of the power of our method, in Section II, we show how color constancy can be achieved using spectral information derived from chromatic aberration.

Section III contains our main result, which is that by applying statistical methods, the surface reflectance can be obtained from the color signal. In Section IV, we study over 2000 applications of the separation algorithm, and we find that the average error in the reflectance component it extracts from the color signal is only about twice that obtainable by least squares fitting of the reflectance basis functions directly to the actual reflectance. For objects described well by the underlying model, we recover the surface spectral reflectance, and hence achieve color constancy, with great accuracy. In Section V, we outline some of the further directions this work is presently taking.

II. COLOR CONSTANCY

This section describes the separation algorithm, how it achieves color constancy, and why this leads to an increase in the dimensionality of the surface reflectance vector space. First, finite-dimensional linear models are reviewed.

A. Finite-Dimensional Models for Illumination and Surface Reflectance

Earlier work on using finite-dimensional linear models in color vision [9], [16] showed that many surface spectral reflectance functions are approximated well by a specific finite-dimensional linear model comprising frequency-limited functions. The idea is the following. A set of spectral reflectance basis functions called $S_j(\lambda)$, with $j$ ranging from 1 to $n$, are selected; then, any surface spectral reflectance is represented as a linear combination of these basis functions. Modeling the surface in this way constitutes a finite-dimensional model for surface spectral reflectance. The basis functions are constrained to being linearly independent. That is, if $\sigma_1S_1(\lambda) + \cdots + \sigma_nS_n(\lambda) = 0$, all the coefficients $\sigma_i$ must be zero. Illumination is modeled by $m$ basis functions, with weights $\epsilon_i$.

The surface spectral reflectance corresponding to point $x$ in an image is expressed as

$$S_x(\lambda) = \sum_{j=1}^{n} \sigma_j S_j(\lambda).$$

Similarly, the spectral power distribution of ambient light expressed in terms of the basis functions is

$$E_x(\lambda) = \sum_{i=1}^{m} \epsilon_i E_i(\lambda).$$

A finite-dimensional model for surface spectral reflectance may not capture the surface characteristics of an object exactly, so we measure the fit by examining the sum-of-squares error formed by the square of the difference between the actual reflectance and the model's approximation of it, integrated over wavelength. Turning the integral into a sum, we consider an error formula

$$\text{error} = \left( \sum_{\lambda} (S_{\text{exact}} - S_{\text{approx}})^2 \right)^{1/2}$$

(cf. [23, p. 177]). The coefficients $\sigma_i$ of the model should be chosen so as to minimize this error.

Over the visible spectrum (400–700 nm), the surface spectral reflectance curves of natural objects are usually reasonably smooth and continuous. Many experiments on empirical surface spectral reflectances show that the surface reflectances of natural objects are very limited, and most of them can be modeled using only a few basis functions. For example, Cohen [18] analyzed the surface spectral reflectances of 150 Munsell chips and computed the characteristic vectors for these chips. His results show that the first three characteristic vectors account for 99.2% of the variance in the fit. Adding another characteristic vector only yields a slight improvement (about 0.5%) [18] (see also [24]).

Maloney [3] showed that the most surface spectral reflectances of natural objects can be modeled adequately by means of three basis functions. Higher dimensions result in better approximations, yet three basis functions still suffice when the filtering effect of the cone response functions is taken into account [5]. Similarly, three–five basis functions are sufficient for modeling most daylight spectra. In any case, our model is essentially unrestricted in terms of the dimensionality of the vector spaces for illumination and reflectance, although, in practice, most of our computations are carried out with both dimensionalities equal to 3.
B. Spectral Reflectance from Edge Information

As a preliminary example, we consider a simple situation in which two regions of different color abut. With a sharp-edged boundary, this situation allows determination of the color signal \( I(\lambda) \) corresponding to the difference between the color signals reflected from the two regions.

At each image location, we assume that there are \( s \) sensor classes corresponding to human cone responses or camera sensitivities. Let the spectral response curves of the sensors be \( R_k(\lambda) \), \( k = 1-s \).

For the color signal,

\[
I'(\lambda) = E' (\lambda) S'(\lambda),
\]

the response \( \rho_k^{x} \) at location \( x \) of sensor class \( k \) can be expressed as the integral over \( \lambda \) of the product of \( R_k(\lambda) \) and the color signal

\[
\rho_k^{x} = \int E'(\lambda) S'(\lambda) R_k(\lambda) \, d\lambda.
\]

Hence, at each pixel in the image, we have

\[
\rho_k^{x} = \int E'(\lambda) S'(\lambda) R_k(\lambda) \, d\lambda = \sum_{i=1}^{m} \sum_{j=1}^{n} \epsilon_i \sigma_j^x g_{ijk}, \quad k = 1-s, \quad \epsilon_i \text{ independent of } x.
\]

In an image in which two regions of different color meet, there will be \( n + s \) variables for the surface spectral reflectances for the two regions and \( m \) variables for the spectral power distribution of the ambient light. Let the two regions be labeled 1 and 2. Then there are \( s + s = 2s \) equations

\[
\rho_k^{x} = \sum_{i=1}^{m} \sum_{j=1}^{n} \epsilon_i \sigma_j^x g_{ijk}, \quad k = 1-s \text{ and } X = 1 \text{ or } 2.
\]

Now, we have shown elsewhere [20]-[22] that the difference \( \Delta(\lambda) \) of the color signals in the two regions can be determined from the chromatic aberration occurring at the edge. Essentially, chromatic aberration at a color edge spatially separates the light in such a way that it can be used to determine the complete difference in color signal for all \( \lambda \).

Hence, from chromatic aberration, we can determine for every \( \lambda \) the set of differences

\[
E(\lambda) S^1(\lambda) - E(\lambda) S^2(\lambda) = \Delta(\lambda)
\]

with \( \Delta(\lambda) \) known.

Substituting the finite-dimensional expansions, we thus have

\[
\left[ \sum_{i=1}^{m} \epsilon_i E_i(\lambda) \right] \left[ \sum_{j=1}^{n} (\sigma_j^1 - \sigma_j^2) S_j(\lambda) \right] = I(\lambda). \tag{2}
\]

Specific values of \( I(\lambda) \) corresponding to specific wavelengths \( \lambda \) can be substituted to obtain as many equations as desired, but only independent equations are actually needed. The dimension of the set of product functions \( E_i(\lambda) S_j(\lambda) \) fixes the number of independent equations for the variables \( \epsilon_i \) and for \( \sigma_j^1, \sigma_j^2 \)—the number of independent equations equaling the dimension of the set. Other methods, such as a statistical approach via a least squares fit, can also be used to derive independent equations for the unknowns \( \epsilon_i \) and \( \sigma_j \).

When all the functions \( E_i(\lambda) S_j(\lambda) \) are linearly independent, we can certainly solve for all \( \epsilon_i(\sigma_j^1 - \sigma_j^2) \) products. Since the unknowns themselves appear as products, however, it is only possible to solve for values of the \( \epsilon_i \) and \( \sigma_j \) relative to each other. Nevertheless, a constraint such as \( \sum_{i=1}^{m} (\epsilon_i)^2 = 1 \) or \( \| \sum_{i=1}^{m} \epsilon_i E_i(\lambda) \| = 1 \) can be used to render the equations solvable, up to an overall multiplicative constant, for all the \( \epsilon_i \) and the differences \( \sigma_j^1 - \sigma_j^2 \). In this case, then, chromatic aberration alone provides \( n \) equations for the variables \( \sigma_j^1 \) and \( \sigma_j^2 \) themselves, as well as providing a complete solution for all the \( \epsilon_i \). So we do not yet have enough equations to solve for all the \( \sigma_j^1 \) and \( \sigma_j^2 \) individually.

In addition to \( \Delta(\lambda) \), however, we also have available the information provided by (1) involving the sensor responses. Each region yields \( s \) equations, so we can make use of our knowledge of the \( \epsilon_i \) and find all the \( \sigma_j \) for region 1, say, provided they are not too numerous. Specifically, for \( n \leq s \), we can solve for all the \( \sigma_j \) in both regions.

If, as in the human visual system, there are three sensors at each point (i.e., \( s = 3 \)), then the dimension of the linear model for surfaces can be as large as 3, subject to the linear independence condition holding. We have tested the first three basis vectors provided by Judd et al. for ambient illumination [17] coupled with the first three basis vectors provided by Cohen for surface reflectance [18], and we found that they do indeed form a set of nine linearly independent product functions.

Being able to model the surface reflectance space with dimensionality 3 represents a distinct improvement on the work of Maloney et al. [3], [1] in that their work permits a dimensionality of only 2 when \( s = 3 \), as in the human visual system. Such a low dimensionality imposes severe limitations when modeling real surfaces.

The above example shows that injecting additional spectral information can result in substantially better color constancy. The reflectance coefficients \( \sigma_j \) obtained are illumination-independent color descriptors. Of course, the improvement is gained at the price of having to measure the color signal difference which, at least in a laboratory
setting, can be gleaned from chromatic aberration [20]-
[22].

We now turn to the general situation in which the com-
plete color signal, not just the color signal difference, is
made available by any measurement method, for ex-
ample, direct spectrometry.

III. SEPARATING ILLUMINATION FROM SURFACE
REFLECTANCE

In this section, we show that if the basis functions for
illumination and those for surface spectral reflectance have
certain properties, it is possible to recover both the illu-
mination and the reflectance from the spectral power dis-
tribution of the incoming light. We then prove a number
of theorems justifying our claim.

A. Exact Models

When both the spectral power distribution of the illu-
mination and the surface spectral reflectance are exactly
represented by finite-dimensional models, they both can
be derived easily and exactly from the color signal, up to
a multiplicative scale factor.

To show this, we start with the product relating illu-
mination, reflectance, and color signal, which is
\[ E(\lambda)S(\lambda) = I(\lambda), \]
and substitute the finite-dimensional model expansions:
\[ I(\lambda) = \sum_{i=1}^{m} \sum_{j=1}^{n} \epsilon_i \epsilon_j E_i(\lambda)S_j(\lambda). \]
If all the product functions \( E_i(\lambda)S_j(\lambda) \) are linearly in-
dependent, then we can choose \( mn \) different \( \lambda \) values \( \lambda_1, \lambda_2, \ldots, \lambda_{mn} \) such that the matrix \( M \), whose columns consist of \( [E_1(\lambda_1)S_1(\lambda_1), E_1(\lambda_2)S_1(\lambda_2), \ldots, E_m(\lambda_{mn})S_j(\lambda_{mn})] \), is nonsingular. Hence, when these \( mn \) values of \( \lambda \) are substituted into the above equations, they can be interpreted as \( mn \) linearly independent equations for the unknowns \( \epsilon_i \) and \( \epsilon_j \) in terms of the known function \( I(\lambda) \) sampled at the wavelengths \( \lambda \). Because of the linear independence property, the solution for the coefficients \( (\epsilon_i, \epsilon_j) \) is unique. If the illumination function \( E(\lambda) \) is normalized\(^2\) such that \( \epsilon_i = 1 \), then we have unique solutions for all \( \epsilon_i \) and \( \epsilon_j \). Thus, it is possible to derive the exact illumination and surface spectral reflectance, except for a scaling factor applied to normalize \( \epsilon_i \) to 1.

B. Approximate Models

In general, finite-dimensional models will not describe
the spectral power distribution of illumination and the sur-
face spectral reflectance exactly, although they should ap-
proximate them well. If
\[ E(\lambda) = \sum_{i=1}^{m} \epsilon_i E_i(\lambda) \]
and
\[ S(\lambda) = \sum_{j=1}^{n} \sigma_j S_j(\lambda) \]

\(^2\text{Wandell [1] and Yuille [7] propose a similar constraint on the } \epsilon \text{ vector.}\)

then we have
\[ I(\lambda) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \epsilon_i \epsilon_j E_i(\lambda)S_j(\lambda). \]

Therefore, the previous method for exact models cannot
be applied. Instead, we claim that a nonlinear least squares
fit will correctly extract the surface reflectance and the
illumination. In particular, we select those coefficients \( \epsilon_i \)
and \( \sigma_j \) that minimize the sum of squares
\[ \sum_{\lambda} \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} \epsilon_i \epsilon_j E_i(\lambda)S_j(\lambda) - I(\lambda) \right]^2 \]

with the condition \( \epsilon_i = 1 \) as above. The solution of this
equation constitutes a nonlinear least squares problem for
illumination and surface reflectance.

In general, one would not necessarily expect a satisfac-
tory fit to be obtained because of the product structure of
the sum. For example, if \( f(\lambda)g(\lambda) \) were the result of a
least squares fit for some function \( I(\lambda) \), then \( f(\lambda)c(\lambda)g(\lambda)c(\lambda) \) would also be a solution for
any nonzero function \( c(\lambda) \). In addition, there may be an
infinite number of solutions to a least squares fit. How-
ever, if the set of possible illumination distributions and
the set of possible surface spectral reflectances each are
modeled well enough individually by a finite-dimensional
model, a least squares fit will provide good results. In the
next section, we prove explicitly the relationship be-
 tween the individual fits and the correct separation of illu-
mination from surface reflectance.

C. Theorems

The separation algorithm derives finite-dimensional ap-
proximations to the actual illumination and reflectance,
and Theorem 2 below expresses how the error in these
approximations varies with the minimum error possible
using a fixed set of basis functions. Before proceeding to
Theorem 2, we first establish an error bound for the best
approximation to a product of two functions \( ES \) where this
best approximation is generated as the product of two ap-
proximations to the original functions. We state
the error in terms of the error bounds for approximations
to \( E \) and \( S \) taken separately.

**Theorem 1:** Let \( F \) and \( G \) be sets of bounded functions
and let \( E(\lambda) \) and \( S(\lambda) \) be two given bounded functions;
let all functions have the same domain. Suppose \( F \) con-
ists of functions \( f(\lambda) \). Let \( f' \) be that function which is
closest to \( E \). Then \( f' \) is that \( f \) which minimizes \( \| f - E \| \).

Let \( e_F \) be this least error:
\[ e_F = \min_{f \in F} \left\{ \| f - E \| \right\} = \| f' - E \|. \]

Similarly, let the function \( g' \) be closest to \( S \), and define
the least error for the approximation to \( S \) as
\[ e_G = \min_{g \in G} \left\{ \| g - S \| \right\} = \| g' - S \|. \]
If $c$ is the maximum norm for all the functions in sets $F$ and $G$ and functions $E(\lambda)$ and $S(\lambda)$, then

$$\min_{f \in F \& g \in G} \| fg - ES \| \leq c(e_F + e_G).$$

The best approximation formed from the best available product functions $fg$ has an error bounded by a bounded multiple of the sum of the individual best approximation errors. In other words, the better each function $E$ and $S$ can be approximated individually, the better the best product function $fg$, chosen from the set of pairwise products of the available approximating functions, will fit $ES$.

The Appendix contains all proofs.

**Corollary 1:** Let $E_i(\lambda)$, $i = 1-m$, $S_j(\lambda)$, $j = 1-n$, $E(\lambda)$, and $S(\lambda)$ be given, bounded functions, and let all functions have the same domain. Let $I = [a, b]$ be a bounded, closed interval. Let $e_F = \min_{e \in E} \left\{ \left\| \sum_{i=1}^{m} e_i E_i - E \right\| \right\}$ and $e_G = \min_{e \in G} \left\{ \left\| \sum_{j=1}^{n} a_j S_j - S \right\| \right\}$. Then there exists some constant $c$ such that

$$\min_{e_i, e_j \in I} \left\{ \left\| \sum_{i=1}^{m} e_i E_i \sum_{j=1}^{n} a_j S_j - ES \right\| \right\} \leq c(e_F + e_G).$$

This corollary states that the error in the least squares approximation to the color signal is proportional to the errors in the individual fits to the illumination and surface spectral reflectance. If the finite-dimensional linear models closely approximate the illumination and surface spectral reflectance, the error in the least squares fit of the product of the illumination and the surface spectral reflectance will be small. Now we adopt the constraint $e_i = 1$ and show a type of converse of the previous theorem. Namely, we show that the coefficients $e_i$ and $a_j$ representing the best fit to a given color signal $E(\lambda)S(\lambda)$ and obtained by solving the nonlinear least squares problem for illumination and surface reflectance, in fact, provide a good fit to $E(\lambda)$ and $S(\lambda)$ individually. Again, the error found is proportional to the errors involved in fitting illumination and surface reflectance separately with the finite-dimensional model.

**Theorem 2:** Let $E_i(\lambda)$, $i = 1-m$, $S_j(\lambda)$, $j = 1-n$, $E(\lambda)$, and $S(\lambda)$ be given, bounded functions, and let all functions have the same domain. Let $I = [a, b]$ be a bounded, closed interval. Let $e_F = \min_{e \in E} \left\{ \left\| \sum_{i=1}^{m} e_i E_i - E \right\| \right\}$ where $e_i = 1$ and let $e_G = \min_{e \in G} \left\{ \left\| \sum_{j=1}^{n} a_j S_j - S \right\| \right\}$. Let $x_i, i = 1-m$ and $s_i = 1$ and let $y_j, j = 1-n$ be those particular values of $e_i$ and $a_j$ such that a best fit is provided for the product function $ES$:

$$\min_{e_1, \ldots, e_m} \left\{ \left\| \sum_{i=1}^{m} x_i E_i \sum_{j=1}^{n} y_j S_j - ES \right\| \right\} = \min_{e_1, \ldots, e_m} \left\{ \left\| \sum_{i=1}^{m} x_i E_i \sum_{j=1}^{n} y_j S_j - ES \right\| \right\}$$

with $e_1 = 1$. If all the product functions $E_i(\lambda)S_j(\lambda)$, $i = 1-m$ and $j = 1-n$ are linearly independent, then there exist constants $C$ and $D$ such that

$$\min_{e_1, \ldots, e_m} \left\{ \left\| \sum_{i=1}^{m} x_i E_i - E \right\| \right\} \leq C(e_F + e_G)$$

and

$$\min_{e_1, \ldots, e_m} \left\{ \left\| \sum_{j=1}^{n} y_j S_j - S \right\| \right\} \leq D(e_F + e_G).$$

See the Appendix for a proof.

**D. Discussion**

From Theorem 2, we know that if the basis functions $E_i(\lambda)$ for the spectral power distribution of the illumination and $S_j(\lambda)$ for the surface spectral reflectance have the property that all $E_i(\lambda)S_j(\lambda)$ are linearly independent, a good approximation for the surface spectral reflectance and illumination can be achieved by using a least squares fit.

As we stated in Section II, when $n = 3$ and $m = 3$, the nine illumination–reflectance, basis-function products do form a linearly independent set. In fact, the linear independence condition can be relaxed; as long as the product functions form a set with dimension not less than the number of unknown variables minus 1, then our method will find the correct solution. In the important $(3, 3)$ case, we do find linear independence so that our analysis can be applied directly, and we can be confident that a good least squares fit will result.

Our scheme for determining illumination and surface spectral reflectance consists of finding the most plausible separation of illumination and surface reflectance from their product. In general, functions $f(\lambda)$ and $g(\lambda)$ cannot be derived from their product $f(\lambda)g(\lambda)$, but this separation can be done if the two functions are very different. The difference between the illumination and the surface spectral reflectance is captured by the linear independence of the set of functions $E_i(\lambda)S_j(\lambda)$. Since Judd’s basis functions for daylight and Cohen’s basis functions for reflectance have the required property, illumination and surface spectral reflectance can be recovered from their product.

**E. Implementation**

We implemented the separation algorithm using the first three of Cohen’s vectors [18] as basis functions for sur-
face spectral reflectance and the first three of Judd's vectors [17] as basis functions for the spectral power distribution of ambient light. If these particular basis functions are used, all $E_i(\lambda) S_j(\lambda)$ products are linearly independent. Color signals are sampled at 10 nm intervals from 400 to 650 nm since published data for sample spectra exist in this range [19].

Given the spectral power distribution of the color signal $I(\lambda)$, the least squares approach consists of minimizing the quantity
\[
\sum_{i=0}^{n} \left[ \sum_{j=1}^{m} \epsilon_j E_i(\lambda_j) \right] \left[ \sum_{j=1}^{m} \sigma_j S_j(\lambda_j) - I(\lambda) \right]^2.
\]

This overdetermined set of equations is to be solved for unique solutions $\epsilon_j$, $\sigma_j$. By the calculus of variations, the corresponding "normal" equations are as follows.

For $k = 1-m$,
\[
\sum_{i=0}^{n} E_i(\lambda_k) \left[ \sum_{j=1}^{m} \sigma_j S_j(\lambda_k) \right] \left[ \sum_{j=1}^{m} \epsilon_j E_i(\lambda_j) \right] - I(\lambda_k) = 0.
\]

For $k = 1-n$,
\[
\sum_{i=0}^{n} S_i(\lambda_k) \left[ \sum_{j=1}^{m} \epsilon_j E_i(\lambda_k) \right] \left[ \sum_{j=1}^{m} \sigma_j S_j(\lambda_k) \right] - I(\lambda_k) = 0.
\]

We began by using the mathematical library package minpack [25] as a solver for this system of nonlinear equations in the unknowns $\epsilon_j$ and $\sigma_j$. Since the daylight spectral power distribution is very limited, we can make a very good guess for the $\epsilon_j$, and hence for the $\sigma_j$, as initial values for the minpack programs. However, the minpack program turns out to be very unstable for this problem. That is, for slightly different initial values, it produces different solutions.

As a result, a new algorithm producing stable results was developed based on the fact that the first $m$ equations above are linear in $\epsilon_j$ and the last $n$ equations are linear in $\sigma_j$. Given an initial value for the $\epsilon_j$, the last $n$ equations will yield the $\sigma_j$. Substituting these $\sigma_j$ into the first $m$ equations generates another set of $\epsilon_j$. Iteration results in convergent, stable solutions.\footnote{In fact, convergence can easily be proved since all the equations are continuous (see, e.g., [26]).}

IV. APPLICATIONS

For a rigorous appraisal of the above theoretical results, tests were conducted using two types of synthetic color signals. The first kind of color signal is generated by multiplying published spectra for ambient daylight [17] by published surface reflectance functions [19]. The second type consists of random perturbations to the spectra in the space of spectra spanned by the basis functions.

Since the original illumination spectrum is free to not have $\epsilon_j = 1$, the separation algorithm's solutions $\sigma_j$ must be scaled before comparing the resulting reflectance function to the original reflectance function. The appropriate scale factor is that which minimizes the least squares error between them.

As a measure of the separation algorithm's accuracy, we compare its results to that of the best fit possible using the basis functions of the underlying finite-dimensional model.

A. Natural Color Signals

We tested the algorithm by combining each of the 370 surface reflectance functions recorded by Krinov [19] with each of the five standard daylight illumination spectra given by Judd et al. [17]. These five daylight spectra were reconstituted from the set of illumination basis functions developed by characteristic vector analysis [17] of spectral distributions of daylight. Of the 370 available Krinov reflectance spectra, we used only those 345 that were complete in the visible range.

The results were very good in that for most samplings, the overall error obtained in recovering the surface reflectance was only about 18%, the error being defined as
\[
\left\{ \sum_{\lambda} (S - S')^2 / \sum_{\lambda} S^2 \right\}^{1/2}
\]

where $S$ is an actual spectrum and $S'$ is the estimate. We generally found the above error measure to be equal to a multiple of $-0.5-0.8$ times the CIELUV color difference $\Delta E$ in three-dimensional color space [27, p. 828]. This overall 18% error must be compared to the direct fit of the underlying finite-dimensional model using three basis functions to the reflectance alone, which had an average error of 10.0%.

Overall, compared to the errors produced by a finite-dimensional model, least squares fit to the $S'(\lambda)$ spectra separately, the errors for the separation algorithm's extraction of surface reflectance were at most a multiple of 4.18 times as large, with a minimum multiple of 1.00. The average ratio of errors is only about 1.95 times the error of the direct least-squares fit for reflectance, so that the error coefficient $D$ in Theorem 2 is about 2. The median ratio was 1.80 with standard deviation 0.55.

In absolute terms, the largest error for any color signal input was 47.4%; and that case was also one of the most poorly fit by the finite-dimensional model directly, with an error of 26.7%. The ratio of the errors was thus 1.78 in this case.

The best fit had an error of 1.9%, compared to a 1.8% error for the direct least-squares fit to reflectance alone, a ratio of 1.04.

As an example, consider Fig. 1. Here, a color signal has been constructed from the surface spectral reflectance for "river valley with meadows" (general view from a
distance of about 3 km; covered with trees and meadows, end of summer) which is Krinov's sample 54, multiplied by Judd's spectrum for 4800 K correlated color temperature daylight.

Fig. 2 shows the original surface reflectance spectrum given by Krinov, along with the separation algorithm's results. The separation error in this case is 15.2%. The figure also shows the result of using the finite-dimensional model to match the reflectance spectrum directly, without taking any illumination into consideration—an error of 8.1%. So the ratio of the error from our method as compared to the ability of the underlying finite-dimensional model to match the reflectance spectrum is only 1.87. Since the direct least squares fit to reflectance alone is as good as one can get using three basis functions, the separation algorithm does very well.

Fig. 3 shows Judd's daylight spectrum, along with the illumination calculated by the separation algorithm. As can be seen, the results are again quite good. The error for the illumination curve derived by the separation algorithm is 12.8%. The worst case was Krinov reflectance 162 illuminated by 4800 K daylight. The results are shown as Fig. 4. The best case was Krinov reflectance 295 illuminated by 7500 K daylight. The results are shown as Fig. 5.

B. Perturbation Spectra

As a further test, we applied the algorithm to a large set of synthetic reflectances derived by taking the sum of the three reflectance basis functions with specific weights and adding random noise in proportions of 1-5%. The separation algorithm's results in recovering these reflectances under a variety of daylight spectra compare favorably to the best fits of the finite-dimensional model to these spectra.

The reflectance functions defined by weights (1, 1, 1) and (1, 2, 1.5) combined with each of the five available daylight illumination spectra generate ten cases. For the synthetic reflectances produced in this way without noise added, the basis function matching, of course, is exact. In accordance with Theorem 2, the separation algorithm also extracts the exact reflectance spectrum in this case because $e_F$ and $e_G$ are both zero.

When a perturbative noise contribution is added, the average error of both methods rises. Overall, the ratio of the error of the separation algorithm in recovering the reflectance spectrum to the error involved in directly matching the reflectance spectrum using the reflectance basis functions was about 1.17.
Fig. 4. Worst case: Krinov reflectance 162 illuminated by 4800 K daylight. Error from separation algorithm: 47.4%. Error using finite-dimensional model to fit reflectance alone: 26.7%. Ratio: 1.78.

Fig. 5. Best case: Krinov reflectance 295 illuminated by 7500 K daylight. Error from separation algorithm: 1.9%. Error using finite-dimensional model to fit reflectance alone: 1.8%. Ratio: 1.04.

Fig. 6 displays the performance of the separation algorithm in comparison to the accuracy of the underlying finite-dimensional model for the color signal of the synthetic reflectance defined by weights (1, 1, 1) combined with 10 000 K daylight. Perturbations from 1 to 5% by 0.1% make up 41 separate sets. From the regression line to the results, one sees that while the errors for both methods rise as the perturbation percentage increases, the ratio of the errors for the two methods stays approximately constant at about 1.25. This means that the separation algorithm does about as well as the finite-dimensional model itself with respect to these synthetic, unphysical spectra over a broad range of perturbations. The maximum ratio was 2.08, while the minimum ratio was 1.02.

A histogram of the test results for 410 color signals (41 perturbations of each reflectance under five illuminations) is shown in Fig. 7. The ratio of the error in the separation algorithm's reflectance estimate compared to direct modeling of the perturbed reflectance with the finite-dimensional model has an average of 1.17, a maximum of 2.08, and a minimum of 1.00. As can be seen from the histogram, most of the ratios lie below the 75% percentile value of 1.2. The standard deviation is 0.183.

V. CONCLUSION

The separation algorithm divides a color signal into two components, one due to surface reflectance, and another due to the incident illumination. As such, it provides a color-constant descriptor of surface reflectance, although it does require the entire color signal, not just a three-receptor sampling of the color signal.
As shown by Theorem 2 and the perturbation tests, the method works well when compared to the finite-dimensional model on which it is based. Needless to say, linear models of low dimensionality will not approximate surface reflectances and illuminant spectra well in every case, however. Theorem 2 relates the error in the finite-dimensional approximations to the true illumination and reflectance to the error in those recovered from the color signal. The perturbation tests explore the method’s sensitivity to noise, and the results indicate that it is relatively stable.

For applications requiring color-constant descriptors, the color signal at a pixel can be easily measured by any of a number of spectrometry methods. It is much more difficult to obtain the color signal spectrum at all pixels. One solution to this problem is to take into account the tendency of illumination to vary slowly spatially in comparison to changes in reflectance. When this is the case, given an accurate reflectance measurement at one pixel, correct reflectances can be propagated to the rest of the image. The propagation method [28] is based on Horn’s lightness algorithm [29], and extends it by use of finite-dimensional models to colored Mondrian images in which the spectral content, not just the intensity, of the illumination is allowed to vary spatially. Color-constant descriptors throughout an image can be calculated, therefore, either by providing the color-signal spectrum at every pixel or by providing it at one or more pixels (the more the better, to avoid error accumulation), applying the separation algorithm, and propagating the resulting reflectances.

Another problem is that because the separation algorithm holds $e_1 = 1$, it determines the spectral reflectance function at each pixel only up to a scaling factor; the brightness at each point is undetermined. Again, the propagation method which injects the $p_k$ receptor information into the calculation and compares adjacent pixels for those where only a brightness change occurs allows the $s_k$ to be scaled relative to each other so that only one overall multiplicative factor is required for the entire image.

Besides traditional spectrometry and chromatic aberration, another way of obtaining enough information about the color-signal spectrum to obtain color constancy is to use mutual reflection [30].

Perhaps the most important finding in the present work is the linear independence of the set of product basis functions $E_i S_j$ formed from the illumination bases of Judd [17] and the reflectance bases of Cohen [18]. This observation seems to capture an underlying physical property of a large sample of illumination and spectral reflectance functions, namely, that they combine into color signals in a way that allows them to be separated back into their original components.

Appendix

First, we state some useful results that are needed later (see, e.g., [31]).

\textbf{Lemma 1:} Let $x_i, y_i, i = 1$–$n$ be real numbers. Then

$$\sum_{i=1}^{n} (x_i y_i) \leq \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} y_i^2 \right)^{1/2}.$$  

This is a well-known theorem. It simply states that the cosine of the angle between two vectors is less than or equal to 1 (Cauchy–Schwarz inequality).

\textbf{Lemma 2:} Triangle inequality:

$$\sum_{i=1}^{n} (x_i + y_i)^2 \leq \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2.$$  

\textbf{Lemma 3:} Define the norm of a function $f$ as

$$\| f \| = \left( \sum_{i=1}^{n} f^2(\lambda_i) \right)^{1/2}$$  

where all the $\lambda_i$ are given distinct values in the domain of the function. Then

$$\| f + g \| \leq \| f \| + \| g \|.$$  

\textbf{Lemma 4:} $\| f - g \| \leq \| f - h \| + \| h - g \|.$

\textbf{Lemma 5:} $\| f - g \| \leq \| f - h \| + \| h - g \|.$

\textbf{Fact:} $\| f \| = \| -f \|.$

\textbf{Lemma 6:} Let $x_i$ and $y_i, i = 1$–$n$ be real numbers. Then

$$\sum_{i=1}^{n} \left( x_i y_i \right) \leq \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} y_i^2 \right)^{1/2}.$$  

\textbf{Proof:} Since $x_i^2 \geq 0$ and $y_i^2 \geq 0,$

$$\sum_{i=1}^{n} x_i^2 \leq \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \sum_{i=1}^{n} y_i^2 \leq \left( \sum_{i=1}^{n} y_i^2 \right)^{1/2} = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \sum_{i=1}^{n} y_i^2 \leq \sum_{i=1}^{n} x_i y_i.$$  

\textbf{Lemma 7:} $\| fg \| \leq \| f \| \times \| g \|.$

With these results, we can address the problem of minimizing the least squares sum.

\textbf{Proof of Theorem 1:} Since $c$ is the bound of the norms for all functions in sets $F$ and $G$ and functions $E(\lambda)$ and $S(\lambda),$ we have $\| f(\lambda) \| \leq c, \| g(\lambda) \| \leq c, \| E(\lambda) \| \leq c,$ and $\| S(\lambda) \| \leq c$ for all $f \in F$ and $g \in G.$ We have defined $f'$ in $F'$ and $g'$ in $G$ as those functions for which $e_r = \| f' - E \|$ and $e_c = \| g' - S \|$ hold. Then letting $f$ and $g$ range over all possibilities, we seek that product $fg$ which best approximates the given product function $ES$. Therefore, consider the least error of such an approximation:

$$\min_{f \in F, g \in G} \| fg - ES \|.$$
since the minimum norm is certainly less than or equal to and for the best fit coefficients $u_i$ and $v_j$, define
\[ f'(\lambda) = \sum_{i=1}^{m} u_i E_i(\lambda) \quad \text{and} \quad g'(\lambda) = \sum_{j=1}^{n} v_j S_j(\lambda). \]

Now consider the least squares problem for the product function $ES$:
\[
\begin{align*}
\| f g - ES \| &\leq \| f g - (f' - P)(g' - Q) \| \\
&= \| f g - f'g' + f'Q + g'P - PQ \| \\
&= \| f g - f'g' - (f'Q - g'P + PQ) \|.
\end{align*}
\]

Since \( \| fg - ES \| \leq c(e_F + e_G) \) for some constant $c$ by Corollary 1, we have
\[
\| fg - f'g' - (f'Q - g'P + PQ) \| \leq c(e_F + e_G).
\]

In addition, \[ \sum_{i=1}^{m} e_i E_i(\lambda) \] are bounded functions. Therefore, there exists a constant $d$ such that
\[ \| \sum_{i=1}^{m} e_i E_i(\lambda) \| \leq d \]
and
\[ \| \sum_{j=1}^{n} a_j S_j(\lambda) \| \leq d. \]

As a result of Lemma 4, we know that
\[
\begin{align*}
\| f g - f'g' \| - \| (-f'Q - g'P + PQ) \| \\
&\leq \| f g - f'g' - (f'Q - g'P + PQ) \| \\
&\leq \| f g - f'g' - (f'Q - g'P + PQ) \| + \| (-f'Q - g'P + PQ) \|.
\end{align*}
\]

Note that for the last term on the right-hand side, we have
\[
\| (-f'Q - g'P + PQ) \| \leq \| -f'Q \| + \| -g'P \| + \| PQ \| \text{ by Lemma 3}
\]
\[ = \| f'Q \| + \| g'P \| + \| PQ \|. \]

Hence, collecting these results, we find that the difference between the best fit to the product and the product of the best fits to the individual functions satisfies
\[
\begin{align*}
\| fg - f'g' \| &\leq c(e_F + e_G) + \| Q' \| \\
&\quad + \| P' \| + \| PQ \| \\
&\quad \leq c(e_F + e_G) + \| Q \| \times \| f' \| + \| P \| \\
&\quad \times \| g' \| + \| P \| \times \| Q \| \\
&\quad \leq c(e_F + e_G) + e_G d + e_F d + e_F e_G \\
&\quad = (c + d)(e_F + e_G) + e_F e_G.
\end{align*}
\]
That is, substituting the coefficient expansions of \(f, g, f', \) and \(g'\),
\[
\left\| \sum_{i=1}^{m} \sum_{j=1}^{n} (x_iy_j - u_iu_j) E_i S_j \right\| \leq (c + d) (e_F + e_G) + e_F e_G.
\]

Let the difference function on the left be called \(H(\lambda)\):
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} (x_iy_j - u_iu_j) E_i S_j = H(\lambda).
\]
Then
\[
\left\| H \right\| \leq (c + d) (e_F + e_G) + e_F e_G
\]
so we have a bound on the difference function.

We can now make use of this bound to develop bounds on the errors involved in modeling \(E(\lambda)\) and \(S(\lambda)\) separately by using the results \(x_i, y_j\) of the nonlinear least squares fit to the product function \(E(\lambda) S(\lambda)\). We shall use the bound on \(H\) to establish the required properties of \(x_i, y_j\) by supposing \(H(\lambda)\) to exist and solving for \((y_j - v_j)\) in terms of it. Then the bound on \(H\) will translate into a bound on \((y_j - v_j)\).

To solve for the coefficients \((x_i, y_j - u_iu_j)\), consider the vector function
\[
V(\lambda) = \left[ E_1(\lambda) S_1(\lambda), E_1(\lambda) S_2(\lambda), \cdots, E_m(\lambda) S_\lambda(\lambda) \right].
\]
The product function \(E_i(\lambda) S_j(\lambda)\) forms the \([(i-1)n + j]\)th element of the vector function \(V(\lambda)\). To see this, substitute \(i = 1\) and \(j = 1\); then \(E_1 S_1\) will be in column \((1-1)n + 1 = 1\) of \(V(\lambda)\). For \(i = m\) and \(j = n\), we have \((i-1)n + j = (m-1)n + n = nm\) so \(E_m S_n\) will be in column \(nm\) of \(V(\lambda)\).

Now choose \(nm\) \(\lambda\) values \(\lambda_1, \cdots, \lambda_{nm}\) so that the \(nm\) vectors \(V(\lambda_1), \cdots, V(\lambda_{nm})\) are linearly independent. Such a choice ensures that the matrix \(M = \left\{ V(\lambda_1), \cdots, V(\lambda_{nm}) \right\}'\) is not singular where \(\{ V_1, \cdots, V_n \}'\) is the column vector transpose of the row vector \(\{ V_1, \cdots, V_n \}\). That this choice is possible follows from the linear independence of the \(E_i S_j\)'s. Substituting the \(nm\) values of \(\lambda_i, i = 1-nm\) into the previous equations for \(H(\lambda)\), we have \(nm\) linearly independent equations, namely,
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i y_j - u_i u_j) E_i(\lambda_i) S_j(\lambda_i) = H(\lambda_i),
\]
\(l = 1-nm\)
or, written as a matrix equation,
\[
MW = A
\]
where \(W\) is the column vector \([x_1y_1 - u_1u_1, x_1y_2 - u_1u_2, \cdots, x_{nm}y_m - u_{nm}u_{nm}]'\) and \(A\) is the column vector \([H(\lambda_1), \cdots, H(\lambda_{nm})]'\).

Let \(M^{-1} = (b_{l,q})\) where \(M^{-1}\) is the inverse matrix of \(M\). Since the \(\lambda_i\) are predetermined, we can easily form the following number (the matrix norm):
\[
b = \sum_{l=1}^{nm} \sum_{q=1}^{nm} |b_{l,q}|,\]
which is a constant.

Now, the system of equations in \(H(\lambda_i)\) is solvable, and after solving this system for the coefficients of the functions \(E_i S_j\), we have
\[
W = M^{-1} A
\]
or
\[
x_i y_j - u_i u_j = \sum_{l=1}^{nm} b_{(i-1)n+j,l} H(\lambda_l),
\]
\(i = 1-m, j = 1-n\).

Taking the absolute value, we have
\[
|y_j - v_j| \leq b \left( (c + d) (e_F + e_G) + e_F e_G \right],
\]
\(j = 1-n\).

Now we introduce the restriction that normalization of the \(\epsilon\) vectors will be determined by \(\epsilon_1 = 1\). Since \(x_1 = u_1 = 1\), the above implies
\[
|y_j - v_j| \leq b \left( (c + d) (e_F + e_G) + e_F e_G \right],
\]
\(j = 1-n\).
Hence, we finally arrive at a restriction on the fit of \(g(\lambda) = \sum_{j=1}^{n} y_j S_j(\lambda)\) to the function \(S(\lambda)\) where the coefficients \(\lambda_i\) are determined from a nonlinear least squares fit to the color signal \(E(\lambda) S(\lambda)\):
\[
\left\| g - S \right\| \leq \left\| g - (g' - Q) \right\| \leq \left\| g - g' \right\| + \left\| Q \right\| = \sum_{j=1}^{n} |y_j - v_j| \left\| S_j \right\| + e_G \leq \sum_{j=1}^{n} \left\| S_j \right\| \left| b \right| (c + d) (e_F + e_G) + e_F e_G + e_F e_G.
\]
From the above inequality, it follows that there exists a constant \(D\) such that \(\| g - S \| \leq D (e_F + e_G)\). And it is straightforward to show that there exists another constant \(C\) for which the inequality \(\| f - E \| \leq C (e_F + e_G)\) holds, so that the theorem is proved.
REFERENCES


