Notes on Satisfiability-Based Problem Solving
FO Semantics – Sample Problems

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1. For all FO formulas \( \phi \) and \( \psi \),
\[ \exists x (\phi \land \psi) \models (\exists x \phi \land \exists x \psi). \]

To show the implication holds, we suppose \( M, \sigma \) are such that \( M \models \exists x (\phi \land \psi)[\sigma] \), and then show that \( M \models (\exists x \phi \land \exists x \psi)[\sigma] \). We do this by applying the definition of the satisfiability relation \( \models \) (several times):
\[
M \models \exists x (\phi \land \psi)[\sigma] \\
\Rightarrow \text{There is some } a \in M \text{ s.t. } M \models (\phi \land \psi)[\sigma(x \mapsto a)] \\
\Rightarrow \text{There is some } a \in M \text{ s.t. } M \models \phi[\sigma(x \mapsto a)] \text{ and } M \models \psi[\sigma(x \mapsto a)] \\
\Rightarrow \text{There is some } a \in M \text{ s.t. } M \models \phi[\sigma(x \mapsto a)] \\
\text{and there is some } a \in M \text{ s.t. } M \models \psi[\sigma(x \mapsto a)] \\
\Rightarrow M \models \exists x \phi[\sigma] \text{ and } M \models \exists x \psi[\sigma] \\
\Rightarrow M \models (\exists x \phi \land \exists x \psi)[\sigma], \text{ as required.}
\]

2. Show there are FO formulas \( \phi \) and \( \psi \) such that
\[ (\exists x \phi \land \exists x \psi) \models \exists x (\phi \land \psi). \]

We show this by exhibiting a counterexample to the implication, in the form of a choice for \( \phi \) and \( \psi \), and a suitable structure. Let \( \phi \) be \( Px \) and \( \psi \) be \( Qx \). We need a structure for vocabulary \( (P, Q) \) that satisfies \( (\exists x \phi \land \exists x \psi) \) but does not satisfy \( \exists x (\phi \land \psi) \). Let \( M \) be such that \( M = \{1, 2\} \), \( P^M = \{1\} \) and \( Q^M = \{2\} \). To verify this suffices, we apply the definition of the satisfaction relation. We have that:
\[
1 \in P^M \text{ and } 2 \in Q^M \\
\Rightarrow \text{There is some } a \in M \text{ s.t. } M \models Px[\sigma(x \mapsto a)] \\
\text{and there is some } a \in M \text{ s.t. } M \models Qx[\sigma(x \mapsto a)], \text{ for any } \sigma \\
\Rightarrow M \models \exists x Px[\sigma] \text{ and } M \models \exists x Qx[\sigma] \\
\Rightarrow M \models (\exists x Px \land \exists x Qx)[\sigma], \text{ as required.}
\]

We also have that:
\[
M \models \exists x (Px \land Qx)[\sigma] \\
\Rightarrow \text{There is some } a \in M \text{ s.t. } M \models (Px \land Qx)[\sigma(x \mapsto a)] \\
\Rightarrow \text{There is some } a \in M \text{ s.t. } M \models Px[\sigma(x \mapsto a)] \text{ and } M \models Qx[\sigma(x \mapsto a)] \\
\Rightarrow \text{There is some } a \in M \text{ s.t. } a \in P^M \text{ and } a \in Q^M \\
\Rightarrow P^M \cap Q^M \neq \emptyset, \text{ which contradicts our choice of } M \text{ (so } M \not\models (\exists x (Px \land Qx)[\sigma])).
\]

3. Show that \( \exists x \forall y PxY \models \forall y \exists x PxY \).

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Suppose \( M, \sigma \) are such that \( M \models \exists x \forall y P xy[\sigma] \). We must show that \( M \models \forall y \exists x P xy[\sigma] \). We have that:

\[
M \models \exists x \forall y P xy[\sigma]
\]
\[
\Rightarrow \text{There is some } a \in M \text{ s.t. } M \models \forall y P xy[\sigma(x \mapsto a)]
\]
\[
\Rightarrow \text{There is some } a \in M \text{ s.t. for every } b \in M, M \models P xy[\sigma(x \mapsto a)(y \mapsto b)]
\]
\[
\Rightarrow \text{There is some } a \in M \text{ s.t. for every } b \in M, (a, b) \in P^M
\]
\[
\Rightarrow \text{For every } b \in M, (a, b) \in P^M, \text{ for a specific } a \text{ chosen to witness the “some } a \text{” in the previous line}
\]
\[
\Rightarrow \text{For every } b \in M, \text{ there is some } a \in M \text{ s.t. } (a, b) \in P^M
\]
\[
\Rightarrow \text{For every } b \in M, \text{ there is some } a \in M \text{ s.t. } M \models P xy[\sigma(y \mapsto b)(x \mapsto a)]
\]
\[
\text{(Notice that, because variables } x \text{ and } y \text{ are distinct, the valuations } [\sigma(y \mapsto b)(x \mapsto a)]
\]
\[
\text{and } [\sigma(x \mapsto a)(y \mapsto b)] \text{ are the same.)}
\]
\[
\Rightarrow \text{For every } b \in M, M \models \exists x P xy[\sigma(y \mapsto b)]
\]
\[
\Rightarrow M \models \forall y \exists x P xy[\sigma], \text{ as required.}
\]

4. Show that, for any FO formula \( \phi \), \( \exists x \forall y \phi \models \forall y \exists x \phi \).

The proof is the same as for the special case, above, but we don’t know the formula \( \phi \), so must modify the lines that use \( P \). Suppose \( M, \sigma \) are such that \( M \models \exists x \forall y \phi[\sigma] \). Let

\[
S = \{ (a, b) \in M^2 \mid M \models \phi[\sigma(x \mapsto a)(y \mapsto b)] \}.
\]

Now,

\[
M \models \exists x \forall y \phi[\sigma]
\]
\[
\Rightarrow \text{There is some } a \in M \text{ s.t. } M \models \forall y \phi[\sigma(x \mapsto a)]
\]
\[
\Rightarrow \text{There is some } a \in M \text{ s.t. for every } b \in M, M \models \phi[\sigma(x \mapsto a)(y \mapsto b)]
\]
\[
\Rightarrow \text{There is some } a \in M \text{ s.t. for every } b \in M, (a, b) \in S
\]
\[
\Rightarrow \text{For every } b \in M, (a, b) \in S, \text{ for } a \text{ chosen to witness the existential in the previous line}
\]
\[
\Rightarrow \text{For every } b \in M, \text{ there is some } a \in M \text{ s.t. } (a, b) \in S
\]
\[
\Rightarrow \text{For every } b \in M, \text{ there is some } a \in M \text{ s.t. } M \models \phi[\sigma(y \mapsto b)(x \mapsto a)]
\]
\[
\Rightarrow \text{For every } b \in M, M \models \exists x \phi[\sigma(y \mapsto b)]
\]
\[
\Rightarrow M \models \forall y \exists x \phi[\sigma], \text{ as required.}
\]

5. Show there are FO formulas \( \phi \) such that \( \forall x \exists y \phi \not\models \exists y \forall x \phi \).

Let \( \phi \) be the formula \( P xy \), and \( M \) be such that \( M = \{1, 2, 3\} \) and \( P^M = \{(1, 1), (2, 2), (3, 3)\} \). We will show that \( M \models \forall x \exists y P xy \), but \( M \not\models \exists y \forall x P xy \). We have that:

For every \( a \in M, (a, a) \in P^M
\]
\[
\Rightarrow \text{For every } a \in M, \text{ there is some } b \in M \text{ s.t. } (a, b) \in P^M
\]
\[
\Rightarrow \text{For every } a \in M, \text{ there is some } b \in M \text{ s.t. } M \models P xy[\sigma(x \mapsto a)(y \mapsto b)] \text{ (for any } \sigma)
\]
\[
\Rightarrow M \models \forall x \exists y P xy
\]

Also,

\[
M \models \exists y \forall x P xy[\sigma]
\]
⇒ There is some \( a \in M \) s.t. for every \( b \in M \), \( \mathcal{M} \models P_{xy}[\sigma(x \mapsto a)(y \mapsto b)] \)
⇒ There is some \( a \in M \) s.t. for every \( b \in M \), \((a, b) \in P^M\)

However, there is no such \( a \in M \), so \( \mathcal{M} \not\models \exists y\forall x P_{xy}[\sigma] \). Therefore, \( \forall x \exists y P_{xy} \not\models \exists y\forall x P_{xy} \).