1. For each of the following statements about FO formulas, state whether the property holds for all FO formulas or not. Justify each answer by either a careful semantic proof based on the definition of the satisfaction relation, or by giving a counterexample with a brief explanation of why it a counterexample.

   (a) For any two FO formulas $\phi$ and $\psi$, $\exists x(\phi \land \psi) \models (\exists x\phi \land \exists x\psi)$.

   (b) For any two FO formulas $\phi$ and $\psi$, $(\exists x\phi \land \exists x\psi) \models \exists x(\phi \land \psi)$.

   (c) For any FO formula $\phi$, $\exists x\forall y\phi \models \forall y\exists x\phi$.

   (d) For any FO formula $\phi$, $\forall x\exists y\phi \models \exists y\forall x\phi$.

2. (a) The formula $\exists x(\exists yx \neq y)$ defines the class of structures with domains of size at least two, and $\exists x\exists y(x \neq y \land \exists z(z = x \lor z = y))$ defines the class of structures with domains of size exactly two.

   (i) Write a formula $A$, containing only the relation symbol $=$ and no function symbols, that defines the class of structures with domain size exactly three.

   (ii) Let $\tau$ be the vocabulary $(S, =)$, where $S$ is a unary relation symbol. Write a $\tau$-formula $A$ such that, for each $\tau$-structure $M$, $M \models A$ if and only if $S^M$ contains exactly 3 elements.

   (b) Write a FO formula defining the class of graphs consisting of exactly one directed triangle.

   (c) Write a FO formula defining the class of graphs consisting of a union of disjoint triangles.

3. A directed Hamiltonian cycle in a directed graph $G = (V, A)$ is a set of arcs in $G$ that makes a simple cycle which visits every vertex of $G$. In such a cycle, every vertex has exactly one in-coming arc and one out-going arc. We can define this property by the following formula $\phi$:

   $$\forall u((\exists!vAvu) \land (\exists!vAu))$$

   The quantifier $\exists!$ is read “there exists a unique”. That is, the formula $\exists!x\phi(x)$ is an abbreviation for the formula $\exists x(\phi(x) \land \forall y(\phi(y) \rightarrow y = x))$.

   Let $\mathcal{K}$ be the class of all directed graphs that have directed Hamiltonian cycles.

   (a) Prove that $\text{Mod}[\phi]$ is not $\mathcal{K}$.

   (b) The Directed Hamiltonian Cycle problem is in NP (it is easy to check if a set of arcs is a Hamiltonian cycle), so there must be an $\exists SO$ formula that defines $\mathcal{K}$. Give such a formula. (Hint: If there is a directed Hamiltonian cycle in $G$, then there is a ordering of the vertices in which there is an arc from each vertex to the next, and one from the last element back to the first. You can also think in terms of a suitable permutation of $V$)