Adjusted Interval Digraphs

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Abstract

Interval digraphs were introduced by West et al. They can be recognized in polynomial time and admit a characterization in terms of incidence matrices. Nevertheless, they do not have a forbidden structure characterization nor a low-degree polynomial recognition algorithm.

We introduce a new class of 'adjusted interval digraphs', obtained by a slight change in the definition. We show that, by contrast, these digraphs have a natural forbidden structure characterization, parallel to a characterization for undirected graphs, and admit a simple recognition algorithm.

We relate adjusted interval digraphs to a list homomorphism problem. Each digraph H defines a corresponding list homomorphism problem L-HOM(H). We observe that if H is an adjusted interval digraph, then the problem L-HOM(H) is polynomial time solvable, and conjecture that for all other reflexive digraphs H the problem L-HOM(H) is NP-complete. We present some preliminary evidence for the conjecture, including a proof for the special case of semi-complete digraphs.

1 Introduction

An *interval graph* [13] is a graph H which admits an *interval representation*, i.e., a family of intervals $I_v, v \in V(H)$, such that $uv \in E(H)$ if and only if I_u and I_v intersect. An *interval digraph* [27] is a digraph H which admits an *interval pair representation*, which is a family of pairs of intervals $I_v, J_v, v \in V(H)$, such that $uv \in E(H)$ if and only if I_u intersects J_v. Note that an interval graph must be reflexive (each vertex has a loop), but an interval digraph may lack loops. If the intervals $I_v, J_v, v \in V(H)$, can be chosen so that for each v the intervals I_v and J_v have the same left endpoint, we say that H is an *adjusted interval digraph*. It is again clear that an adjusted interval digraph must be reflexive.

In [3] we have studied the special case of adjusted interval digraphs H representable by intervals $I_v, J_v, v \in V(H)$, in which each interval J_v is just one point. These are called *chronological interval digraphs* [3], and we have shown that they can be characterized by the absence of certain special forbidden structures. In [26], a related class of *interval catch* *digraphs* has been characterized by the absence of certain other forbidden structures. Here we provide a forbidden structure characterization of adjusted interval digraphs, directly implying a simple polynomial time recognition algorithm for the class of adjusted interval digraphs. The forbidden structure we introduce, an *invertible pair*, is also useful in a number of similar situations [18, 17]. In particular, for the class of interval graphs [18], the resulting characterization is equivalent to the well known characterizations in terms of induced cycles and asteroidal triples [23], or in terms of consecutive clique enumerations [12]. We note that the class of interval digraphs [27] so far lacks an elegant forbidden structure characterization, and the only algorithm for their recognition to date is a dynamic programming algorithm of complexity $O(nm^6(n+m)\log n)$ [24].

We apply adjusted interval digraphs to the complexity of list homomorphisms. A *homomorphism* f of a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ in which $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$ [20]. If $L(v), v \in V(G)$, are *lists* (subsets of $V(H)$), then a *list homomorphism* of G to H (with respect to the lists L) is a homomorphism satisfying $f(v) \in L(v)$ for all $v \in V(G)$. The *list homomorphism problem* $L-HOM(H)$ asks whether or not an input digraph G equipped with lists L admits a list homomorphism $f : G \to H$ with respect to L. The complexity of the list homomorphism problem L-HOM(H) for undirected graphs H has been classified in [5, 6, 7]. Of particular interest for this paper is the classification in the special case of reflexive graphs: if H is a reflexive graph, then the problem L-HOM(H) is polynomial time solvable if H is an interval graph, and is NP-complete otherwise [5]. The complexity of $L-HOM(H)$ for any digraph (and more general relational system) has been classified in [1] (see Theorem 4.1). For reflexive digraphs H, we propose a simpler classification. Specifically, we observe that each adjusted interval digraph H has polynomial time solvable list homomorphism problem L-HOM(H), and conjecture that for any other reflexive digraph H the problem L-HOM(H) is NP-complete. We offer some evidence for the conjecture here, including a proof for the case of semi-complete digraphs. We have some additional evidence for the conjecture, including the case of oriented trees, and we hope to be able to completely prove the conjecture in the full version of this paper.

2 Invertible Pairs

The *underlying graph* of H has an edge uv whenever $uv \in E(H)$ or $vu \in E(H)$. If u, v are adjacent in the underlying graph of H, the pair uv is a *forward edge* if $uv \in E(H)$, and a *backward edge* if $vu \in E(H)$. Note that a loop is both a forward edge and a backward edge. If $uv \in E(H)$, we say that u *dominates* v in H.

We define two walks $P = x_0, x_1, \ldots, x_n$ and $Q = y_0, y_1, \ldots, y_n$ in H to be *congruent*, if they follow the same pattern of forward and backward edges, i.e., if x_ix_{i+1} is a forward (backward) edge if and only if y_iy_{i+1} is a forward (backward) edge, respectively. If P and Q as above are congruent walks, we say that P *avoids* Q, if there is no edge x_iy_{i+1} in the same direction (forward or backward) as $x_i x_{i+1}$.

An *invertible pair* in H is a pair of vertices u, v such that

- there are congruent walks P from u to v and Q from v to u such that P avoids Q,
- there are congruent walks P' from v to u and Q' from u to v such that P' avoids Q'.

It will turn out to be useful to reformulate these definitions in terms of an auxiliary digraph. The *pair-digraph* H^+ associated with H has vertices $V(H^+) = \{(u, v) : u \neq v\}$, and edges $(u, v)(u', v')$, where

$$uu', vv' \in E(H) \text{ and } uv' \notin E(H), \text{ or}$$
$$u'u, v'v \in E(H) \text{ and } v'u \notin E(H).$$

Lemma 2.1 *If H has an invertible pair (u, v), then (u, v) and (v, u) belong to the same strong component C of the pair-digraph H^+; moreover, for any (x, y) in C the reversed pair (y, x) also belongs to C, i.e., each pair in C is invertible.*

If H has no invertible pair, then for each strong component C of H^+ there exists a reversed strong component *C' such that $(x, y) \in C$ if and only if $(y, x) \in C'$.*

An ordering $<$ of the vertices of H is a *min ordering* of H if it satisfies the following property: if $uv \in E(H)$ and $u'v' \in E(H)$, then $\min(u, u') \min(v, v') \in E(H)$. (A min ordering was also called an *X-underbar enumeration* [14, 20]). The following result relates min orderings to adjusted interval digraphs.

Theorem 2.2 *A reflexive digraph is an adjusted interval digraph if and only if it admits a min ordering.*

Min orderings also play an important role for list homomorphism problems, cf. [20].

Theorem 2.3 *[14] If H admits a min ordering, then the problem $L - HOM(H)$ is polynomial time solvable.*

Finally, we observe that an invertible pair is an obstruction to the existence of a min ordering.

Lemma 2.4 *If H has an invertible pair, then H does not admit a min ordering.*

Proof. Suppose $(u, v)(u', v')$ is an edge of the pair-digraph H^+. Suppose $<$ is a min ordering of H, and suppose $u < v$. The we must also have $u' < v'$. Following the directed closed walk in H^+ which contains (u, v) and (v, u), we obtain a contradiction. □

3 Adjusted Interval Digraphs

We now strengthen Lemma 2.4.

Theorem 3.1 *A reflexive digraph H admits a min ordering if and only if it has no invertible pair.*

In fact, we shall prove the following stronger result.

Theorem 3.2 *The following statements are equivalent for a reflexive digraph H:*

1. *H is an adjusted interval digraph*
2. *H has a min ordering*
3. *H has no invertible pairs*
4. *The vertices of H^+ can be partitioned into sets D, D' such that*
   - $(x, y) \in D$ *if and only if* $(y, x) \in D'$
   - $(x, y) \in D$ *and* (x, y) *dominates* (x', y') *in* H^+ *implies* $(x', y') \in D$
   - $(x, y), (y, z) \in D$ *implies* $(x, z) \in D$.

Proof. The equivalence of 1 and 2 is proved in Theorem 2.2. Furthermore, Lemma 2.4 shows that 2 implies 3. It is also quite straightforward to see that 4 implies 2; it suffices to define $a < b$ if $(x, y) \in D$. Thus it remains to show that 3 implies 4.

Therefore, we assume that H has no invertible pair. Note that we may assume that H is weakly connected, otherwise we can order each weak component separately. We also note that for each strong component C of H^+, there is a corresponding reversed strong component C' whose pairs are precisely the reversed pairs of the pairs in C; we shall say that C, C' are *coupled* strong components.

The partition of $V(H^+)$ into D, D' will correspond to separating each pair of coupled strong components C, C' of H^+. The vertices of one strong components will be placed in the set D, their reversed pairs will go to D'. We wish to make these choices in such a way as to avoid creating a *circular chain* in D, i.e., a sequence of pairs $(x_0, x_1), (x_1, x_2), \ldots,$ $(x_n, x_0) \in D$.

We shall proceed as follows. Initially the sets D and D' are empty. We say that a strong component C of H^+ is *ripe* when it has no edge *to* another strong component in $H^+ - D$. In the general step, we shall take a ripe component C and place it in D, and simultaneously place C' in D'. (Note that C' need not be ripe, but has no edge *from* another strong component.) We will show that there is always at least one ripe strong component which can be added to D without creating a circular chain.

The sets D, D' will always have the following properties (which are true initially). There is no circular chain in D; each strong component of H^+ belongs entirely to D, D', or to $V(H^+) - D - D'$; the pairs in D' are precisely the reversed pairs of the pairs in D; there is no edge of H^+ from D to a vertex outside of D; and there is no edge of H^+ from a vertex outside of D' to a vertex in D'. At the end of the algorithm each pair (x, y) with $x \neq y$ will belong either to D or to D', and hence the final D will have no circular chain and hence satisfy the transitivity property of 4.

It remains to prove that the algorithm maintains these properties. This proof will appear in the full version of the paper. □

This gives us a polynomially verifiable forbidden subgraph characterization of adjusted interval digraphs. As noted above, checking for invertible pairs amounts to computing the strong components of H^+ and checking for the existence of a pair $(u, v), (v, u)$ in one strong component. Thus the major work is the computation of the auxiliary graph H^+.

Corollary 3.3 *Let H be a reflexive digraph. Then H is an adjusted interval digraph if and only if it has no invertible pair.* ◇

4 Polymorphisms

The min orderings defined above are a particular case of the following general concept. Let k be a positive integer. The *k-th power* of H is the digraph H^k with vertex set $V(H)^k$ in which $(u_1, u_2, \ldots, u_k)(v_1, v_2, \ldots, v_k)$ is an edge just if each $u_i v_i$ is an edge of H. A *polymorphism of order k* is a homomorphism of H^k to H. A polymorphism f is *conservative* if $f(u_1, u_2, \ldots, u_k)$ always is one of $u_1, u_2, \ldots, u_k$. From now on we shall use the word *polymorphism to mean a conservative polymorphism.*

A polymorphism f of order two is *commutative* if $f(u, v) = f(v, u)$ for any u, v. If H admits a min ordering $<$, then clearly defining $f(u, v) = \min(u, v)$ is a polymorphism, which is commutative.

Two ternary polymorphisms also play a role in the problems L-HOM(H) [1]. A polymorphism $f : H^3 \to H$ is called a *majority polymorphism* if $f(u, u, v) = f(u, v, u) = f(v, u, u) = u$ for any u, v. A ternary polymorphism $f : H^3 \to H$ is called a *Maltsev polymorphism* if $f(u, u, v) = f(v, u, u) = v$ for any u, v. A ternary polymorphism $f : H^3 \to H$ is *majority (respectively Maltsev) over a, b*, if $f(a, a, b) = f(a, b, a) = f(b, a, a) = a, f(b, b, a) = f(b, a, b) = f(a, b, b) = b$ (respectively if $f(a, a, b) = f(b, a, a) = b, f(a, b, b) = f(b, b, a) = a$).

At this point, we can state the classification of L-HOM(H) due to Bulatov. The theorem applies to any relational structure H, but for our purposes we only need to state it for reflexive digraphs. Recall that by our definition each polymorphism is conservative. Also, we formulate the result in a language of binary commutative polymorphisms in place of the more usual semi-lattice operations [1], since it is equivalent and is more convenient in our context.

Theorem 4.1 *[1] Let H be a reflexive digraph.*

If for every pair of vertices a, b of H there exists a polymorphism of H which is either ternary and majority, or Maltsev, over a, b, or is binary and commutative over a, b, then L-HOM(H) is polynomial time solvable.

Otherwise, if some pair of vertices a, b does not admit any of these polymorphisms, then the problem L-HOM(H) is NP-complete.

5 List Homomorphism Problems

The following fact follows directly from Theorems 2.3 (or Theorem 4.1) and 2.2.

Theorem 5.1 *If H is an adjusted interval digraph, then L-HOM(H) is polynomial time solvable.*

Here is an equivalent form of the conjecture from [9, 15].

Conjecture 5.2 *If H is an adjusted interval digraph, then L-HOM(H) is polynomial time solvable.*

(We also had a similar conjecture for irreflexive digraphs [9, 15]. However, that conjecture has turned out to be false [16, 2], and we shall discuss the case of irreflexive digraphs in a companion paper [16].)

We now provide some preliminary evidence to support our conjecture. In the full version of the paper we will offer additional results to this end. A digraph is semi-complete if its underlying graph is complete.

Theorem 5.3 *Suppose H is a reflexive semi-complete digraph. If H contains an invertible pair, then L-HOM(H) is NP-complete.*

In the proof, we will appeal to Bulatov's characterization, Theorem 4.1, showing that if there exist invertible pairs in H, then some invertible pair a, b admits no polymorphism as prescribed by Theorem 4.1. In the proof, we shall use the following observations. Let R be the reflexive digraph $V(R) = \{a, b, c\}$ and $E(R) = \{aa, bb, cc, ab, bc, ac, ca\}$.

Lemma 5.4 *There is no polymorphism g on the digraph R which is a majority over a, b.*

Lemma 5.5 *Suppose H is a reflexive digraph with $ab \in E(H), ba \notin E(H)$. There is no polymorphism h over the digraph H which is a Maltsev operation over a, b.*

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