Musical Effects of the Digital Pressure Controlled Valve

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Avian Syrinx Model

The Clarinet Reed Model
Outline

• Fletcher’s generalized pressure-controlled valve
• Our generalized model
• Developing the general differential equations for reed displacement and volume flow
• Discretization
• Results using model implemented in Pd
Classification of the Pressure-Controlled Valve

Fletcher uses the couplet \((\sigma_1, \sigma_2)\) to specify the upstream and downstream behaviour of the valve in the presence of additional pressure.

\[\sigma_{1,2} = +1: \text{the valve opens further with a pressure increase}\]

\[\sigma_{1,2} = -1: \text{the valve closes further with a pressure increase}\]
1. 

\((-\), +\) The valve is \textbf{blown closed} as in woodwind instrument reeds or reed pipes of the pipe organ.

2. 

\((+, -\) The valve is \textbf{blown open} as in simple lip-reed models and the human larynx

3. 

\((+, +\) The \textbf{swinging door} or transverse model as in the avian syrinx
Valve displacement:

\[ \frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 (x - x_0) = \frac{1}{m} (\sigma_1 p_0 S_1 + \sigma_2 p_1 S_2) + \frac{1}{m} p_1 S_3 \]
Generalized Model
Valve Displacement

\[ m \frac{d^2 \theta}{dt^2} + m 2\gamma \frac{d\theta}{dt} + k(\theta - \theta_0) = F \]

Fundamental frequency: \( \omega = \sqrt{\frac{k}{m}} \)
The net force acting on the valve

\[ F_m = \omega \lambda_m p_m \quad \quad F_b = -\omega \lambda_b p_b \]

\[ F_U = -\omega \mu \left( p_m - \frac{\rho}{2} \left( \frac{U}{A} \right)^2 \right) \]

\[ F = F_m + F_b + F_U \]
Discretization

\[ m \frac{d^2 \theta}{dt^2} + m2\gamma \frac{d\theta}{dt} + k(\theta - \theta_0) = F \]

Laplace Transform:

\[ ms^2 \Theta(s) + mgs \Theta(s) + k \Theta(s) - k\theta_0 = F(s) \]

Bilinear Transform:

Defined by

\[ s \rightarrow c \left( \frac{1 - z^{-1}}{1 + z^{-1}} \right) \]

where \( c = \frac{2}{T} \)
Transfer function:

\[
\frac{\Theta(s)}{F(s) + k\theta_0} = \frac{1 + 2z^{-1} + z^{-2}}{a_0 + a_1 + a_2z^{-2}} = \frac{1}{a_0} \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{a_1}{a_0}z^{-1} + \frac{a_2}{a_0}z^{-2}}
\]

\[
a_0 = mc^2 + mgc + k
\]
\[
a_1 = -2(mc^2 - k)
\]
\[
a_2 = mc^2 - mgc + k.
\]

Difference equation:

\[
\theta[n] = \frac{1}{a_0} (F_k[n] + 2F_k[n - 1] + F_k[n - 2]) - \frac{a_1}{a_0} \theta[n - 1] - \frac{a_2}{a_0} \theta[n - 2]
\]
Volume Flow

The force on a thin slice $dy$ along the reed is given by

$$F = A(y; x) \Delta p(y)$$

The force is applied to a mass of

$$m = \rho A(y; x) dy$$

Newton’s 2$^{nd}$ law $F = ma$ gives

$$A(y; x) \Delta p(y) = \rho A(y; x) dy \frac{dv}{dt}$$
\[ A(y; x) \Delta p(y) = \rho A(y; x) dy \frac{dv}{dt} \]

Since volume flow is equal to particle velocity scaled by area,

\[ \Delta p(y) = \rho \frac{dU}{dt} dy / A(y; x) \]

Integrate of the length of the channel to obtain

\[ p(0) - p(\mu) = \rho \frac{dU}{dt} \int_{y=0}^{y=\mu} dy / A(y; x) \]

(by Bernoulli’s equation) equal to \( p_b \)
The differential equation governing volume flow is given by

\[
\frac{dU}{dt} = (p_m - p_b) \frac{A(x)}{\nu \rho} - \frac{U^2}{2\nu A(x)}
\]
Feathering the Reed

Recall the differential equation governing airflow:

\[
\frac{dU}{dt} = (\rho_m - \rho_b) \frac{A(x)}{\nu \rho} - \frac{U^2}{2\nu A(x)}
\]
Volume flow approximation when area is small

\[
\frac{dU}{dt} \approx -\frac{U^2}{2\nu A(t)}
\]

is non-linear in \(U(t)\)

Substitute

\[
W(t) = \frac{1}{U(t)}
\]

New differential equation for flow

\[
\frac{dU}{dt} = -\frac{1}{W(t)^2} \frac{dW}{dt}
\]

where

\[
\frac{dW}{dt} = \frac{1}{2\nu A(t)}
\]
Integrate to solve for volume flow

\[ U(t) = \left[ \frac{1}{U(t_0)} + \frac{1}{2\nu A(t_0)}(t - t_0) \right]^{-1} \]

Backwards difference approximation

\[ \frac{dU}{dt} = \frac{U(t_0 + T) - U(t_0)}{T} \]

Small area solution

\[ \frac{dU}{dt} = -\frac{U(t_0)^2}{2\nu A(t_0) + U(t_0)T} \]
Small area approximation

\[ \frac{dU}{dt} \approx - \frac{U^2}{2\nu A(t)} \]

Small area solution

\[ \frac{dU}{dt} = - \frac{U(t_0)^2}{2\nu A(t_0) + U(t_0)T} \]

Leaky Term

The final feathered differential equation for volume flow

\[ \frac{dU}{dt} = (p_m - p_b) \frac{A(t_0)}{\nu \rho} - \frac{U(t_0)^2}{2\nu A(t_0) + U(t_0)T} \]
Results

Blown Closed (pm = 55, fr = 619, fb = 220)

Valve Displacement (cm)

Volume Flow (cm³/s)

Time (s)
Implementation in Pd