Sinusoids (10 points)

1. (3 points.) From the figure below, determine the values for amplitude \( A \), initial phase \( \phi \), and radian frequency \( \omega_0 \) of the sinusoid \( x(t) = A \cos(\omega_0 t + \phi) \).

![Sinusoid Graph](image)

**Solution:**

The cosine function is shifted either to the left by \( \pi/2 \) (\( \phi = \pi/2 \)) or to the right by \( 3\pi/2 \) (\( \phi = -3\pi/2 \)). Since we usually constrain the phase to be between \(-\pi\) and \(\pi\), and at \( t = 0 \) the phase equals \( \pi/2 \) radians, we would typically say the cosine function has an initial phase of \( \phi = \pi/2 \).

Since the time to complete one cycle is 2 seconds, the frequency (in Hz) is given by \( f_0 = 1/2 \) and the radian frequency is given by \( \omega_0 = 2\pi f_0 = \pi \).

The peak amplitude is given by \( A = 3 \).

2. (7 points.) Every single sinusoid can also be expressed as the sum of a sine and cosine function, or equivalently, an “in-phase” and “quadrature-phase” component. That is,

\[
A \cos(\omega_0 t + \phi) = A \cos \phi \cos(\omega_0 t) - A \sin \phi \sin(\omega_0 t) = B \cos(\omega_0 t) + C \sin(\omega_0 t),
\]

where \( B = A \cos \phi \) and \( C = -A \sin \phi \). Prove this result using Euler’s formula and the fact that \( A \cos(\omega_0 t + \phi) \) is the real part of the complex exponential \( Ae^{j(\omega_0 t + \phi)} \).

**Solution:**

\[
A \cos(\omega_0 t + \phi) = \text{Re}\{ Ae^{j(\omega_0 t + \phi)} \} = \text{Re}\{ Ae^{j\omega_0 t} e^{j\phi} \} = \text{Re}\{ A [\cos(\omega_0 t) + j \sin(\omega_0 t)] [\cos \phi + j \sin \phi] \} = \text{Re}\{ A [\cos \phi \cos(\omega_0 t) + j \sin \phi \cos(\omega_0 t) + j \cos \phi \sin(\omega_0 t) - \sin \phi \sin(\omega_0 t)] \} = A \cos \phi \cos(\omega_0 t) - A \sin \phi \sin(\omega_0 t) = B \cos(\omega_0 t) + C \sin(\omega_0 t).
\]
Digital Audio (10 pts)

3. (3 points.) In order to properly represent a continuous-time signal $x(t)$ using a computer, the analog-to-digital converter (ADC) must perform two (2) processes on two (2) continuous variables. What are the processes and corresponding variable being discretized in each case?

Solution:
- Sampling: discretizes time variable $t$.
- Quantization: discretizes instantaneous amplitude $x(t)$.

4. (6 points.) Given a sampling rate $f_s = 1/T_s$, where $T_s$ is the sampling period, there is an infinite number of sinusoids that will give the same sequence as $x(n) = A \cos(2\pi f_0 n T_s + \phi)$. That is, there are infinite aliases that are indistinguishable from $x(n)$.

- Sketch the magnitude of the spectrum of $x(n)$ for $f_0 = f_s/4$, over the range $-2f_s$ to $2f_s$. Include all aliases for both negative and positive frequencies.
- On the same plot and over the same range, sketch the magnitude of the spectrum of $x(n)$ for $f_0 = 5f_s/8$. Use a broken line to distinguish this spectrum from that of the previous sinusoid.
- If the input to a digital audio system is a sinusoid with a frequency $f_0 = 5f_s/8$ Hz, what will the output frequency be in terms of $f_s$?

Solution:

The output frequency of a digital system having an input sinusoid with frequency $f_0 = 5f_s/8$, is $3f_s/8$.

5. (4 points.) The radian frequency of the sinusoid in the figure below is $\omega = \pi/2$ radians. Determine:
   (a) the frequency in Hz
   (b) the duration (in seconds) of the segment shown in the figure
   (c) the sampling rate

Solution:

(a) Frequency in Hz:
\[ f = \frac{\omega}{2\pi} = \frac{\pi}{4\pi} = \frac{1}{4} \text{ Hz} \]

(b) Duration in seconds: If $f = 1/4$ Hz, then first 1/4 cycle has a duration of 1 second. Since the segment shown consists of eight 1/4 cycles, the duration is 8 seconds. Another way: since $f = 1/4$ Hz, the period is $T = 1/f = 4$ seconds and there are two cycles yielding a duration of $2T = 8$ seconds.

(c) Since there are 8 samples in 1 second (or 1/4 cycle), $f_s = 8$. 

\[ f_s = 8 \]
Synthesis (10 points)

6. (10 points.) A Frequency Modulation (FM) synthesizer, having the parameters in the table below, produces an output signal $y(n)$. Using the plot of the Bessel functions below, sketch the magnitude of the spectrum of $y(n)$ over the frequency range 0 to 1700 Hz.

(Note: in your sketches leading to your final answer, you may need to consider frequencies below DC. Carrier and modulating frequencies were chosen so that phase will not effect your final result.)

You should come as close as you can to the amplitudes given by the Bessel functions, and your plot should clearly show any “missing” harmonics.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>carrier frequency</td>
<td>$f_c$</td>
<td>300 Hz</td>
</tr>
<tr>
<td>modulating frequency</td>
<td>$f_m$</td>
<td>250 Hz</td>
</tr>
<tr>
<td>index of modulation</td>
<td>$I$</td>
<td>2</td>
</tr>
</tbody>
</table>

![Bessel functions](Figure 1: Bessel functions of the first kind $J_k$, for orders $k = 0 - 5$.)

Solution:
If $I = 2$, we know that the highest order sideband that has significant amplitude is given by $k = I + 1 = 3$ (or perhaps as much as $k = 4$ since that’s just a rule of thumb).

We have a carrier frequency at $f_c = 300$ Hz, and then components at $f_c \pm f_m$, $f_c \pm 2f_m$, $f_c \pm 3f_m$, and $f_c \pm 4f_m$, producing “preliminary” and “actual” frequency components found in the first two columns of the table below.

The corresponding amplitudes are found by looking at the the Bessel functions of orders 0-4 (provided), where we obtain approximately -0.22, 0.58, 0.35, 0.13, and 0.03 respectively, when indexed with $I = 2$.

Recall, the amplitudes of the even numbered lower sidebands are given by the amplitudes of the corresponding upper sideband, and the amplitudes of the odd numbered lower sidebands are given by $J_{-k}(I) = -J_k(I)$.

In this case, folding over the DC axis produces no interference and thus the phase of the carrier has no consequence on the resulting magnitude spectrum (and it does not matter whether our carrier is a sine wave or a cosine wave).

<table>
<thead>
<tr>
<th>“preliminary” frequency</th>
<th>actual frequency</th>
<th>amplitude</th>
<th>magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>-700</td>
<td>700</td>
<td>$J_{-4}(3) = .1$</td>
<td>.03</td>
</tr>
<tr>
<td>-450</td>
<td>450</td>
<td>$J_{-3}(3) = -.3$</td>
<td>.13</td>
</tr>
<tr>
<td>-200</td>
<td>200</td>
<td>$J_{-2}(3) = .5$</td>
<td>.35</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>$J_{-1}(3) = -.35$</td>
<td>.58</td>
</tr>
<tr>
<td>300</td>
<td>300</td>
<td>$J_0(3) = -.25$</td>
<td>.22</td>
</tr>
<tr>
<td>800</td>
<td>550</td>
<td>$J_1(3) = .35$</td>
<td>.58</td>
</tr>
<tr>
<td>1100</td>
<td>800</td>
<td>$J_2(3) = .5$</td>
<td>.35</td>
</tr>
<tr>
<td>1400</td>
<td>1050</td>
<td>$J_3(3) = .3$</td>
<td>.13</td>
</tr>
<tr>
<td>1700</td>
<td>1300</td>
<td>$J_4(3) = .1$</td>
<td>.03</td>
</tr>
</tbody>
</table>

Plotted below is the actual result calculated in Matlab.