Max- and Min-Neighborhood Monopolies*

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Abstract

Given a graph $G = (V, E)$ and a set of vertices $M \subseteq V$, a vertex $v \in V$ is said to be *controlled* by M if the majority of v's neighbors (including itself) belongs to M. M is called a *monopoly* in Gif every vertex $v \in V$ is controlled by M. For a specified M and a given range for edge set E ($E_1 \subseteq E \subseteq E_2$), we try to determine an E such that M is a monopoly in $G = (V, E)$. We first present a polynomial algorithm for testing if such an E exists, by formulating it as a network flow problem. Assuming that a solution for E does exist, we then show that solutions with the maximum and minimum $|E|$, respectively, can be found in polynomial time, by solving weighted matching problems.

In case there is no solution for E, we want to maximize the number of vertices controlled by the given M. Unfortunately, this problem turns out to be NP-hard. We, therefore, design a simple approximation algorithm which guarantees an approximation ratio of 2.

1 Preliminary

Let $G = (V, E)$ be an undirected graph, where V (resp., E) is the vertex (resp., edge) set. We assume that G is simple, i.e., G contains neither self-loops nor parallel edges. For a vertex $v \in V$, let us define the *neighborhood* of v by $N_G(v) = \{v\} \cup \{w \mid (w, v) \in E\}$. A vertex $v \in V$ is said to be *controlled by* a vertex set $M \subseteq V$ if the majority of its neighbors are in M; i.e.,

$$|N_G(v) \cap M| \quad \geq \quad |N_G(v)|/2. \tag{1}$$

Here we use a non-strict majority (including equality). However all results obtained in this paper hold for the strict majority as well. For a vertex set $M \subseteq V$, let $Cont(G, M)$ denote the set of vertices of G controlled by M. We call the vertex set M a *monopoly* if it controls every vertex in the graph G, i.e., $Cont(G, M) = V$.

The notion of monopoly was introduced by N. Linial et al. [8] to understand local majority voting in a distributed environment. Local majority voting is motivated, for example, by agreement problems in agent systems (e.g., [2, 13, 14]). Let us consider algorithms for the agents to agree on one industrial standard from among many proposed candidate standards [13]. Under the assumption that every agent

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knows who support which proposals, an agreement could be reached by, for example, adopting the candidate standard that the majority of the agents support. The agent system, however, may be spread over a wide geographical area, so that this approach may not be practical. To overcome this problem, heuristic algorithms have been suggested, based on partial (local) information about the distribution of the agents' opinions.

For simplicity, suppose that there are only two proposals, 0 and 1, and that the candidate standard that the majority of the agents support is to be selected as the standard. If each agent knows the opinions of his/her neighboring agents, a natural heuristic to approximate the agreement would be to take the majority opinions of his/her neighbors and oneself. This is called the *deterministic local majority polling system*.

Peleg and his colleagues recently investigated such a system and determined how many agents supporting 0 are necessary and sufficient for the agreement to result in 0 [1, 8, 11]. They model the system by an undirected graph $G = (V, E)$, where V and E represent the set of agents and the neighborhood relation, respectively. Now, we can easily see that all agents decide on 0 if and only if there is a monopoly M in G whose members all support 0. In the deterministic local majority polling systems, securing the support by the members of a monopoly M implies securing unanimous agreement. and therefore, monopolies play an important role in such systems. Some other applications are discussed in [11].

Linial et al. [8] discussed the problems related to monopolies as packing and covering problems for graphs. They showed that $|M| = \Omega(\sqrt{n})$ and gave a graph with a monopoly M of size $O(\sqrt{n})$, where $n = |V|$. As for computational complexity, Peleg [11] showed that the problem of computing a minimum monopoly is NP-hard, by reducing the minimum dominating set problem to it. Based on the non-approximable results [3, 9] on the set cover problem and its variants, including the minimum dominating set problem, the following conjecture is plausible: For any real $\varepsilon > 0$, the minimum monopoly problem has no $(\ln n - \varepsilon)$ approximation unless $NP \subseteq Dtime(n^{\log\log n})$. On the other hand, it is known [11] that a greedy algorithm yields a $(\ln |E| + 1)$ approximation for the minimum monopoly problem. Bermond and Peleg [1] studied some of its modifications, "r-monopoly" and "self-ignoring" monopoly. Repetitive versions of the local majority polling system have also been discussed by several authors [7, 10, 12].

As mentioned above, one can "control" everybody by controlling a small monopoly. This is an interesting property, and would be useful for some applications (e.g., see the papers by D. Peleg). We are thus interested in finding a (smart) way of "controlling" a given set of objects by modifying the system topology. With this motivatation, in this paper, we first consider the following problem:

MONOPOLY VERIFICATION

Input:	Two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, and a vertex set $M \subseteq V$.
Question:	Does there exist a graph $G = (V, E)$ such that (1) $E_1 \subseteq E \subseteq E_2$ and (2) M is a monopoly in G ?

If the answer to the monopoly verification problem is a Yes, we want to compute a graph G with some additional properties. Among such properties, we consider the maximality and minimality.

MAX-NEIGHBORHOOD MONOPOLY

Input:	Two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, and a vertex set $M \subseteq V$.
Output:	A graph $G = (V, E)$ such that (1) $E_1 \subseteq E \subseteq E_2$, (2) M is a monopoly in G, and (3) M is not a monopoly in $G' = (V, E')$ with $E_1 \subseteq E' \subseteq E_2$ and $|E'| > |E|$, if such a G exists; No, otherwise.

Min-Neighborhood Monopoly	
Input:	Two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, and a vertex set $M \subseteq V$.
Output:	A graph $G = (V, E)$ such that (1) $E_1 \subseteq E \subseteq E_2$, (2) M is a monopoly in G, and (3) M is not a monopoly in $G' = (V, E')$ with $E_1 \subseteq E' \subseteq E_2$ and $\|E'\| < \|E\|$, if such a G exists; No, otherwise.

The max-neighborhood monopoly (resp., min-neighborhood monopoly) problem can be interpreted as follows. Assume that the current system topology is represented by $G_2 = (V, E_2)$ (resp., $G_1 = (V, E_1)$). Then the members of M try to find a minimum-cost set of links in $E_2 - E_1$ so that the topology obtained from G_2 (resp., G_1) by removing (resp., adding) such links secures the adoption of the proposal by M, if the members of M pay for the cost of breaking (resp., establishing) links in $E_2 - E_1$.

We note that, if $E_1 = \emptyset$, then the max-neighborhood monopoly problem is a maximum subgraph problem (or a minimum edge-deletion problem). On the other hand, if G_2 is complete (i.e., $G_2 = K_n$, where $n = |V|$), the min-neighborhood monopoly problem is a minimum augmentation problem.

We will then consider the case in which the answer to the monopoly verification problem is No. In this case, we want to compute a graph G in which M controls as many vertices in V as possible.

Max Controlled Set	
Input:	Two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, and a vertex set $M \subseteq V$.
Output:	A graph G such that (1) $E_1 \subseteq E \subseteq E_2$ and (2) $\|Cont(G, M)\| \geq \|Cont(G', M)\|$ holds for all $G' = (V, E')$ with $E_1 \subseteq E' \subseteq E_2$.

Section 2 studies the monopoly verification problem. We show that the problem is polynomially solvable by reducing it to the network flow problem. In Sections 3 and 4, we consider the max- and min-neighborhood monopoly problems, respectively. Although both problems are more general than the monopoly verification problem, we show that they are also polynomially solvable. Section 5 then investigates the max controlled set problem. Unlike the previous problems, it turns out to be intractable, even if either G_1 is empty or G_2 is complete. In the second part of Section 5, we present a simple approximation algorithm which guarantees an approximation ratio of 2.

2 Monopoly Verification Problem

For a graph $G = (V, E)$ and $A, B \subseteq V$, consider an edge set E', which may be a superset of E. Define $E'(A, B) = (A \times B) \cap E' = \{(v, w) \in E' \mid v \in A, w \in B\}$.

If $A = \{v\}$ (resp., $B = \{w\}$), then we simply write $E(v, B)$ (resp., $E(A, w)$) instead of $E(v, B)$ (resp., $E(A, w)$).

Suppose that M is a monopoly in graph $G = (V, E)$. Let $U = V \setminus M$ and $D = E_2 \setminus E_1$, and $D(A, B) = (A \times B) \cap D$. We consider the following two modifications to E; (1) adding edges in $D(M, M) = (M \times M) \cap D$ to E and (2) deleting edges in $D(U, U)$ from E. Since these modifications do not affect the answer to the problem we assume in this section that the edge set E satisfies

$$E \supseteq E_1 \cup D(M, M), \quad (2)$$
$$E \subseteq E_1 \cup D(M, M) \cup D(U, M). \quad (3)$$

For a vertex $v \in U$, let

$$\mathit{deficit}(v) \;=\; |N_{G_1}(v) \cap U| - |N_{G_1}(v) \cap M|. \tag{4}$$

By definition, v is controlled by M in G_1 if and only if $\mathit{deficit}(v) \leq 0$. Let $U_>$ and $U_\leq$ be the sets of vertices $v \in U$ such that $\mathit{deficit}(v) > 0$ and $\mathit{deficit}(v) \leq 0$, respectively. Then $v \in U_\leq$ is controlled by M in any graph G with $E(\supseteq E_1)$ that satisfies (3). Thus we can restrict E to satisfy

$$E(U_\leq, M) \;=\; E_1(U_\leq, M) \qquad (\text{i.e., } E \;\subseteq\; E_1 \cup D(M, M) \cup D(U_>, M)). \tag{5}$$

Let $G^+ = (V, E_1 \cup D(M, M))$, and for a vertex $v \in M$, let

$$\mathit{surplus}(v) \;=\; |N_{G^+}(v) \cap M| - |N_{G^+}(v) \cap U|. \tag{6}$$

By property (2), $\mathit{surplus}(v)$ represents an upper bound on the number of edges $(v, w) \in D(v, U_>)$ that can be added to G_1. If $\mathit{surplus}(v) < 0$ holds for some v, we can see that v cannot be controlled by M in any graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$. To investigate the case where there is a solution for E, we thus assume that all vertices $v \in M$ satisfy $\mathit{surplus}(v) \geq 0$.

We now define a network $N = (G^* = (V^*, E^*), c)$, where $c : E^* \mapsto \mathbb{R}^+$ is a function.

$$\begin{aligned} V^* &= U_> \cup M \cup \{s, t\}, \\ E^* &= E_s \cup E_t \cup D(U_>, M), \end{aligned}$$

where

$$\begin{aligned} E_s &= \{(s, v) \,|\, v \in M\}, \\ E_t &= \{(w, t) \,|\, w \in U_>\}, \end{aligned}$$

and the capacity function

$$c(e) \;=\; \begin{cases} \mathit{surplus}(v) & \text{if } e = (s, v) \in E_s \\ \mathit{deficit}(w) & \text{if } e = (w, t) \in E_t \\ 1 & \text{if } e = (v, w) \in D(U_>, M). \end{cases}$$

For example, let us consider the problem instance given in Figure 1. The corresponding network N is shown in Figure 2.

The following lemma shows the problem under consideration can be reduced to a network flow problem in G^*.

Lemma 1 *There exists a graph $G = (V, E)$ such that (1) $E_1 \subseteq E \subseteq E_2$ and (2) M is a monopoly in G if and only if the network N has a maximum s-t flow whose value is $\sum_{w \in U_>} \mathit{deficit}(w)$.*

Proof. Since $\sum_{w \in V^*} c(w, t) = \sum_{w \in U_>} \mathit{deficit}(w)$, no s-t flow has a value greater than $\sum_{w \in U_>} \mathit{deficit}(w)$.

Let us assume first that the network N has a maximum s-t flow whose value is $\sum_{w \in U_>} \mathit{deficit}(w)$. Since $c(e)$, $e \in E^*$, is an integer, N has an integral maximum s-t flow f; i.e., for each $e \in E^*$, $f(e)$ is an integer. Let

$$E \;=\; E_1 \;\cup\; D(M, M) \;\cup\; \{e \in D(U_>, M) \,|\, f(e) = 1\}.$$

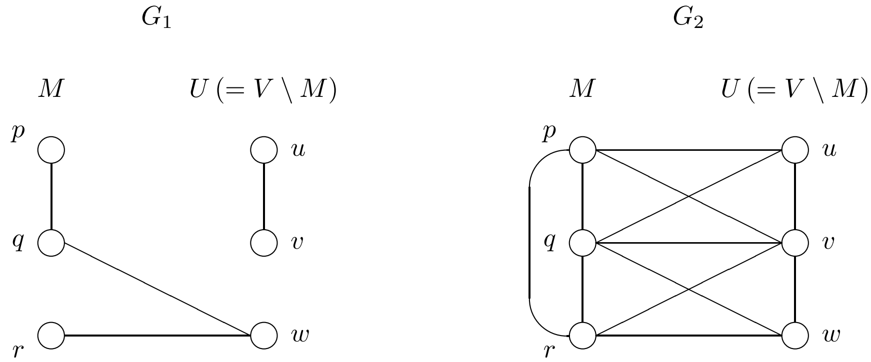

Figure 1: Two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$, where $V = \{p, q, r, u, v, w\}$ and $M = \{p, q, r\}$.

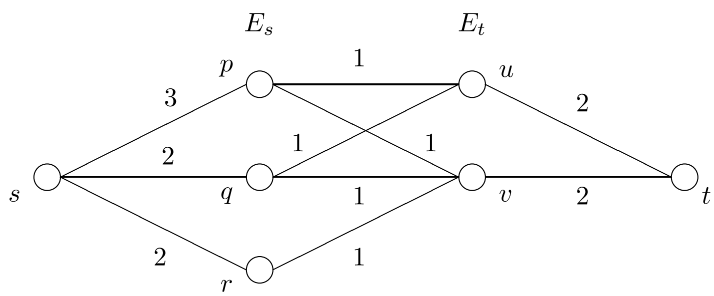

Figure 2: The network $N = (G^* = (V^*, E^*), c : E^* \mapsto \mathbb{R}^+)$ associated with G_1, G_2 and M in Figure 1.

Since f eliminates all deficits, M is a monopoly in the graph $G = (V, E)$.

To prove the only-if part, let us assume that M is a monopoly in the graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$. For each $w \in U_>$, let $\Delta(w)$ be the set of *deficit*(w) arbitrarily chosen edges from $E(w, M) \setminus E_1(w, M)$. Note that M is a monopoly in $G' = (V, E')$, where $E' = E_1 \cup D(M, M) \cup \bigcup_{w \in U_>} \Delta(w)$. We now assign nonnegative integers to each $e \in E^* = E_s \cup E_t \cup D(U_>, M)$ as follows.

$$f(e) = \begin{cases} c^*(e) & \text{if } e = (s, v) \in E_s, \\ \mathit{deficit}(w) & \text{if } e = (w, t) \in E_t, \\ 1 & \text{if } e = (v, w) \in D(M, U_>) \cap \Delta(w), \\ 0 & \text{if } e = (v, w) \in D(M, U_>) \setminus \Delta(w), \end{cases}$$

where $c^*(e) = |\{(v, w) \in \Delta(w) \mid w \in U_>\}|$ for $e = (s, v) \in E_s$. It is clear that this f is an s-t flow and its value is $\sum_{w \in U_>} \mathit{deficit}(w)$. □

For example, Figure 3 shows a maximum flow in the network N given in Figure 2. This flow corresponds to the graph G in Figure 3.

Note that the network N satisfies $|V^*| \leq n + 2$, $|E^*| \leq n + m_2$ and $\max c(e) \leq n$, where $n = |V|$ and

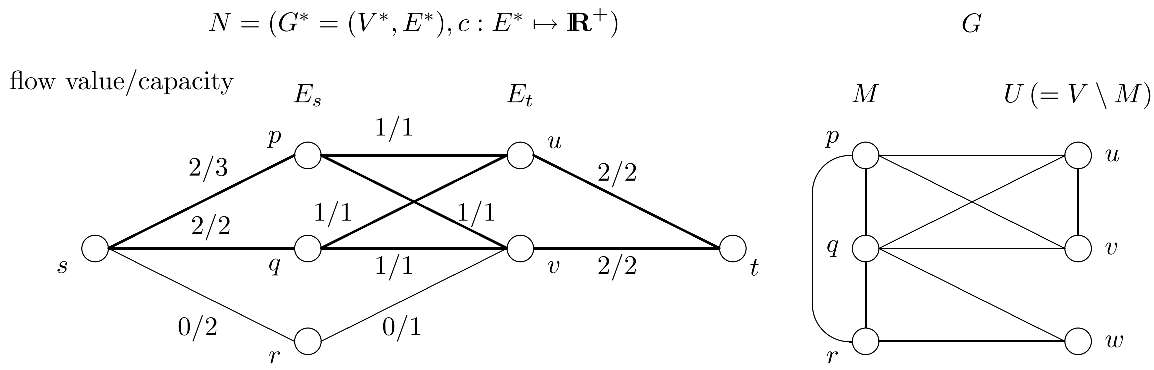

Figure 3: (a) A maximum flow in the network $N = (G^* = (V^*, E^*), c)$, where $c : E^* \mapsto \mathbb{R}^+$, given in Figure 2 and (b) the corresponding graph G.

$m_2 = |E_2|$. Since a maximum flow in such a network can be computed in $\tilde{O}(\min\{(n+m_2)^{3/2}, n^{2/3}(n+m_2)\})$ time [6][1], we have the following result.

Theorem 1 *The monopoly verification problem can be solved in $\tilde{O}(\min\{(n+m_2)^{3/2}, n^{2/3}(n+m_2)\})$ time.* □

3 Max-Neighborhood Monopoly Problem

We assume in this section that the answer to the monopoly verification problem is a Yes, i.e., there exists a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ and M is a monopoly in G. In this case, we show that the max-neighborhood monopoly problem can be solved in polynomial time by finding a maximum weighted matching in an associated graph.

Recall the definitions, $U = V \setminus M$ and $D = E_2 \setminus E_1$. Let $G = (V, E)$ be a solution to the max-neighborhood monopoly problem, where $E = E_1 \cup \Delta$ with $\Delta \subseteq D$. By the maximality of Δ, Δ clearly contains $D(M, M)$. Let $U_>$ and $U_\leq$ be as defined in the previous section. For a vertex $v \in U$ (resp., $v \in M$), define *deficit*(v) (resp., *surplus*(v)) by (4) (resp., (6)). Further, for each $v \in M$, we define the "usable surplus,"

$$surplus^*(v) = \min\{surplus(v), |D(v, U)|\}. \tag{7}$$

By our assumption, $surplus^*(v) \geq 0$ and $|\Delta(v, U)| = surplus^*(v)$ hold for each $v \in M$. For every maximum edge set Δ, $|\Delta(M, M \cup U)|$ is the same, and hence a maximum Δ contains a maximum $\Delta(U, U)$.

We now associate a graph $G^* = (V^*, E^*)$ with $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ and $M \subseteq V$.

$$V^* = V_1 \cup V_2 \cup V_3 \cup V_4,$$
$$E^* = \bigcup_{(v,w) \in D(M \cup U, U)} E_{(v,w)},$$

[1] $\tilde{O}(\,)$ notation is similar to the usual $O(\,)$ notation, except that $\tilde{O}(\,)$ ignores logarithmic factors.

where

$$
\begin{aligned}
V_1 &= \{v_1, v_2, \ldots, v_{surplus^*(v)} \mid v \in M\}, \\
V_2 &= \{v_1, v_2, \ldots, v_{|deficit(v)|} \mid v \in U\}, \\
V_3 &= \{x_{e1}, x_{e2}, x_{e3}, x_{e4} \mid e \in D(M,U)\}, \\
V_4 &= \{z_{<v,w>}, z_{<w,v>} \mid (v,w) \in D(U,U)\}, \\
E_{(v,w)} &= \begin{cases}
\{(v_i, x_{e1}), (x_{e\ell}, x_{e(\ell+1)}), (x_{e2}, w_j) \mid v_i \in V_1, w_j \in V_2, \ell = 1,2,3\} & \text{if } e = (v,w) \in D(M, U_>), \\
\{(v_i, x_{e1}), (x_{e\ell}, x_{e(\ell+1)}) \mid v_i \in V_1, \ell = 1,2,3\} & \text{if } e = (v,w) \in D(M, U_\leq), \\
\{(x_{e4}, z_{<v,w>}), (z_{<v,w>}, z_{<w,v>}), (z_{<w,v>}, x_{e'4}) \mid e \in D(M,v), & \\
\qquad e' \in D(M,w)\} & \text{if } (v,w) \in D(U_>, U_>), \\
\{(x_{e4}, z_{<v,w>}), (v_i, z_{<v,w>}), (z_{<v,w>}, z_{<w,v>}), (z_{<w,v>}, x_{e'4}) \mid & \\
\qquad e \in D(M,v), e' \in D(M,w), v_i \in V_2\} & \text{if } (v,w) \in D(U_\leq, U_>), \\
\{(x_{e4}, z_{<v,w>}), (v_i, z_{<v,w>}), (z_{<v,w>}, z_{<w,v>}), (z_{<w,v>}, x_{e'4}), & \\
\qquad (z_{<v,w>}, w_j) \mid e \in D(M,v), e' \in D(M,w), v_i, w_j \in V_2\} & \text{if } (v,w) \in D(U_\leq, U_\leq).
\end{cases}
\end{aligned}
$$

Here $z_{<v,w>}$ and $z_{<w,v>}$ are new vertices, and we assume that $z_{<v,w>} \neq z_{<w,v>}$. Further, let us define a function $weight : E^* \mapsto \mathbb{R}^+$ by

$$
weight(e^*) = \begin{cases}
4L & \text{if } e^* = (x_{e1}, x_{e2}), \\
L & \text{if } e^* = (x_{e3}, x_{e4}), \\
3L & \text{if } e^* \neq (x_{e1}, x_{e2}), (x_{e3}, x_{e4}) \text{ and } e^* \in E_{(v,w)} \text{ with } (v,w) \in D(M,U), \\
3 & \text{if } e^* = (z_{<v,w>}, z_{<w,v>}), \\
2 & \text{otherwise,}
\end{cases} \tag{8}
$$

where

$$
L > \sum_{e \in D(U,U)} weight(E_e). \tag{9}
$$

Here $weight(T) = \sum_{e^* \in T} weight(e^*)$ for any set $T \subseteq E^*$.

For example, Figure 4 shows the graph G^* associated with the problem instance given in Figure 1.

Note that for every $(v,w) \in D(M \cup U, U)$, $E_{(v,w)}$ forms a tree, and satisfies $E_{(v,w)} \cap E_{(v',w')} = \emptyset$ if $(v,w) \neq (v',w')$. By (8) and (9), we have

$$
weight(e^*) > \sum_{(v,w) \in D(U,U)} weight(E_{(v,w)}) \tag{10}
$$

for each $e^* \in E_{(v,w)}$ with $(v,w) \in D(M,U)$.

We now show that a maximum weighted matching S in G^* corresponds to a desired graph G. Let *Weight* denote the weight of a maximum weighted matching S, i.e., $Weight = weight(S)$. Let $\Theta = \Theta_1 + \Theta_2$, where

$$
\Theta_1 = L \sum_{v \in M} surplus^*(v) + L \sum_{v \in U_>} deficit(v) + 5L|D(M,U)| \tag{11}
$$

$$
\Theta_2 = 3|D(U,U)| + |\Delta(U,U)|. \tag{12}
$$

(Recall that $G = (V, E = E_1 \cup \Delta)$ is a desired graph.)

Lemma 2 *Weight* $\geq \Theta$ *holds.*

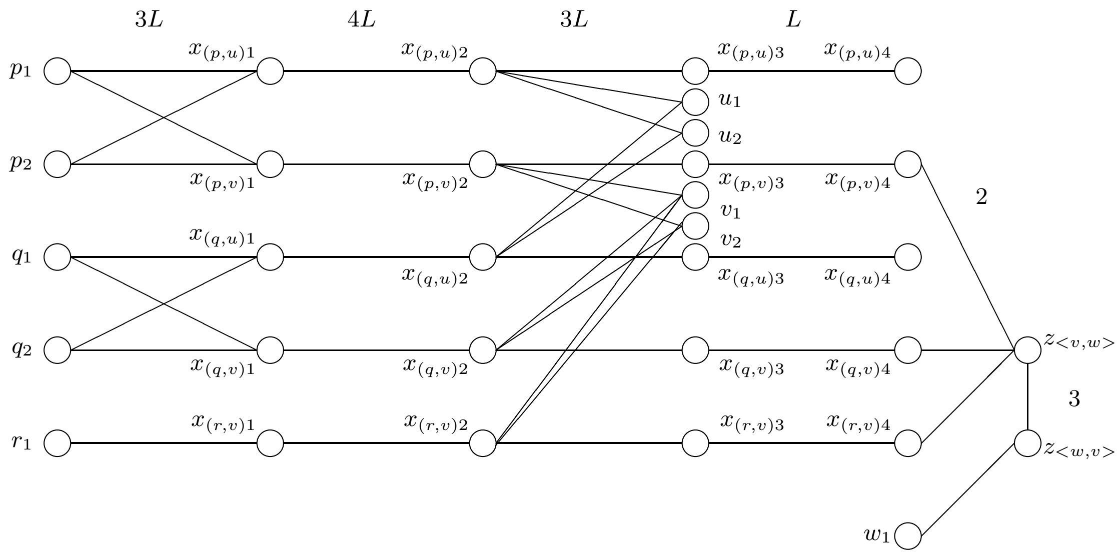

Figure 4: The graph $G^* = (V^*, E^*)$ and weight, *weight* : $E^* \mapsto \mathbb{R}^+$, associated with G_1, G_2 and M in Figure 1.

Proof. Let us construct from Δ a matching T in G^* with weight $\mathit{weight}(T) = \Theta$. We first introduce mappings α and β for each edge $(v, w) \in \Delta(M, U)$.

Let α be an arbitrary one-to-one mapping from $\Delta(M, U)$ to V_1 such that $\alpha(v, w) = v_i$ for some $i \in \{1, 2, \ldots, \mathit{surplus}^*(v)\}$. Since every $v \in M$ satisfies $|\Delta(v, U)| = \mathit{surplus}^*(v)$, α is well-defined. Let $V' = \{w_1, w_2, \ldots, w_{\mathit{deficit}(w)} \mid w \in U_>\}$ and $V'' = \{x_{e3} \mid e \in D(M, U)\}$. Let β be an arbitrary injective mapping from $\Delta(M, U)$ to $V' \cup V''$ such that (i) $\beta(v, w) = w_i$ or $x_{(v,w)3}$ for every $(v, w) \in \Delta(M, U)$, and (ii) for every $w_i \in V'$, there exists an edge $(v, w) \in \Delta(M, U)$ such that $\beta(v, w) = w_i$. Here a mapping χ is called *injective* if $\chi(p) \neq \chi(q)$ for $p \neq q$. Since M is a monopoly in G, β is also well-defined. These α and β show how to allocate the surplus on M to the deficit on U, where $\beta(v, w) = x_{(v,w)3}$ means that (v, w) produces a surplus on w.

Similarly, for each $w \in U$, let γ_w be an injective mapping from $\Delta(w, U)$ to $\{w_1, w_2, \ldots, w_{-\mathit{deficit}(w)}\} \cup \{x_{(v,w)4} \mid \beta(v, w) = x_{(v,w)3}\}$, where $\{w_1, w_2, \ldots, w_{-\mathit{deficit}(w)}\} = \emptyset$ if $\mathit{deficit}(w) \geq 0$. Intuitively, $\gamma_w(w, u) = w_i$ means that the surplus on w is used to add (w, u) to G_1. On the other hand, $\gamma_w(w, u) = x_{(v,w)4}$ means that the surplus on v which is transferred through the edge (v, w) to w is used to add (w, u) to G_1.

From these mappings, we define below a matching T in G^*. For an edge $e = (v, w) \in D(M, U)$, define a set T_e of edges in G^* by

$$T_e = \begin{cases} \{(\alpha(e), x_{e1}), (x_{e2}, \beta(e)), (x_{e3}, x_{e4})\} & \text{if } e \in \Delta(M, U) \text{ and } \beta(e) = w_i \text{ for some } i, \\ \{(\alpha(e), x_{e1}), (x_{e2}, \beta(e))\} & \text{if } e \in \Delta(M, U) \text{ and } \beta(e) = x_{e3}, \\ \{(x_{e1}, x_{e2}), (x_{e3}, x_{e4})\} & \text{otherwise,} \end{cases} \quad (13)$$

and for an edge $e = (w, u) \in D(U, U)$, define a set T_e of edges in G^* by

$$T_e = \begin{cases} \{(\gamma_w(e), z_{<w,u>}), (\gamma_u(e), z_{<u,w>})\} & \text{if } e \in \Delta(U, U), \\ \{(z_{<w,u>}), z_{<u,w>})\} & \text{otherwise.} \end{cases} \quad (14)$$

Let $T = \bigcup_{e\in D(M\cup U,U)} T_e$. Note that T forms a matching in G^*. Note also that, for an edge $e = (v,w) \in D(M,U)$, $\mathit{weight}(T_e) = 7L, 6L$ or $5L$, which respectively correspond to the first, second, and third cases in (13). Exactly $\sum_{w\in U_>} \mathit{deficit}(w)$ (resp., $\sum_{v\in M} \mathit{surplus}^*(v) - \sum_{w\in U_>} \mathit{deficit}(w)$ and $|D(M,U)| - \sum_{v\in M} \mathit{surplus}^*(v)$) edges belong to the first case (resp., the second and third cases). Thus we have

$$\sum_{e\in D(M,U)} \mathit{weight}(T_e) \quad = \quad \Theta_1. \tag{15}$$

For an edge $e = (w,u) \in D(U,U)$, $\mathit{weight}(T_e) = 4$ or 3, which correspond to the first and second cases in (14), respectively. Exactly $|\Delta(U,U)|$ (resp., $|D(U,U)| - |\Delta(U,U)|$) edges belong to the first case (resp., the second case). Thus, we have

$$\sum_{e\in D(U,U)} \mathit{weight}(T_e) \quad = \quad \Theta_2, \tag{16}$$

which together with (15) implies that $\mathit{weight}(T) = \Theta$. This completes the proof. □

To prove the opposite inequality to Lemma 2, we first show the following two lemmas.

Lemma 3 *Let T be a matching in G^* such that $\mathit{weight}(T) \geq \Theta_1$. Then* (15) *holds.*

Proof. For each edge $e \in D(M,U)$, E_e forms a tree, and T_e is a matching in the subgraph (V, E_e), where $T_e = T \cap E_e$. The weight of a maximum matching in $E_{(v,w)}$ is $5L$ when $\mathit{surplus}^*(v) = 0$. When $\mathit{surplus}^*(v) > 0$, the weight is $7L$ if $\mathit{deficit}(w) > 0$, and $6L$, otherwise. However, since T is a matching, (1) among those edges (v,w) with $\mathit{surplus}^*(v) > 0$ and $\mathit{deficit}(w) > 0$, at most $(\sum_{w\in U_>} \mathit{deficit}(w) T_{(v,w)})$ edges can have weight $7L$, and (2) among those with $\mathit{surplus}^*(v) > 0$, at most $(\sum_{v\in M} \mathit{surplus}^*(v) T_{(v,w)})$ edges can have weight at least $6L$. Since the weight of T_e is at least $5L$, we have

$$\sum_{e\in D(M,U)} \mathit{weight}(T_e) \quad \leq \quad L \sum_{v\in M} \mathit{surplus}^*(v) + L \sum_{v\in U_>} \mathit{deficit}(v) + 5L|D(M,U)|. \tag{17}$$

Moreover, if $\sum_{e\in D(M,U)} \mathit{weight}(T_e) < \Theta_1$, then

$$\sum_{e\in D(M,U)} \mathit{weight}(T_e) \quad \leq \quad \Theta_1 - L. \tag{18}$$

By (10), this implies (15). □

Lemma 4 *Weight $\leq \Theta$ holds.*

Let T be a matching in G^* such that $\mathit{weight}(T) \geq \Theta_1$. Then the proof of Lemma 3 also implies that $\bigcup_{e\in D(M,U)} T_e$ gives a desirable Δ' in the sense that M is a monopoly in $G = (V, E_1 \cup D(M,M) \cup \Delta')$, where Δ' is obtained by reversing the construction of (13). Moreover, this implies that $\bigcup_{e\in D(U,U)} T_e$ gives a desirable Δ'' (i.e., M is a monopoly in $G = (V, E_1 \cup D(M,M) \cup \Delta' \cup \Delta''))$, where Δ'' is obtained by reversing the construction of (14). More precisely, $e \in \Delta''$ if and only if $\mathit{weight}(T_e) = 4$. This proves the lemma. □

From Lemmas 2 and 4, we obtain an interesting characterization of the max-neighborhood monopoly problem.

Corollary 1 *Weight* $= \Theta$ *holds.* □

For example, Figure 5 shows a maximum weighted matching in G^* given in Figure 4. In fact, this matching corresponds to G_2, which is clearly a desired graph.

Let us note that graph G^* has sizes $|V^*| = O(m_2)$, $|E^*| = O(m_2^2)$ and $\max weight(e^*) = O(m_2^2)$, where $m_2 = |E_2|$. Since a maximum weighted matching in such a graph can be computed in $\tilde{O}(m_2^{5/2})$ time [4], we have the following theorem.

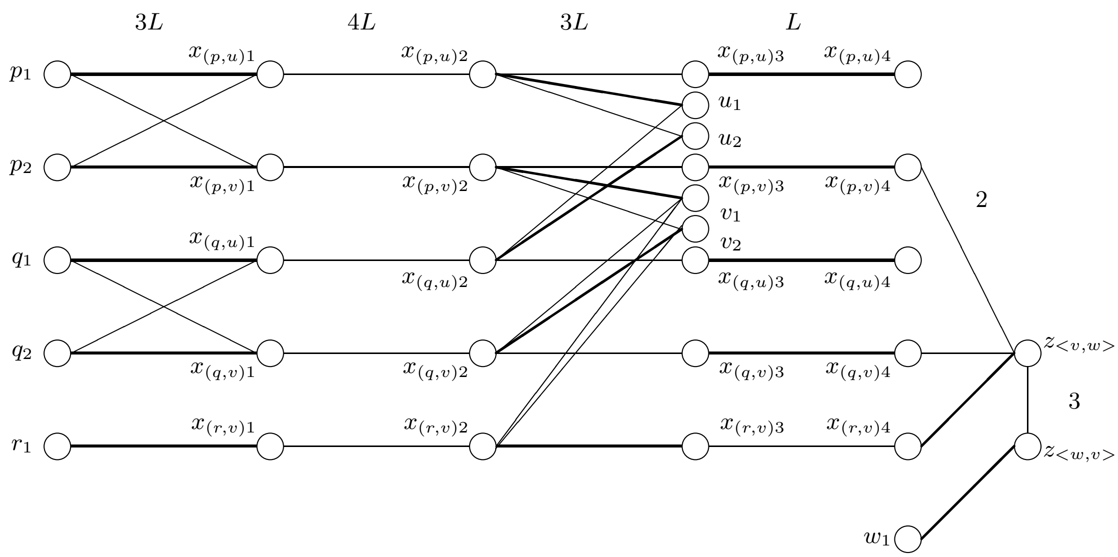

Figure 5: A maximum weighted matching in $G^* = (V^*, E^*)$ given in Figure 4.

Theorem 2 *The max-neighborhood monopoly problem can be solved in* $\tilde{O}(m_2^{5/2})$ *time.* □

4 Min-Neighborhood Monopoly Problem

As in Section 3, we assume in this section that there exists a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ and M is a monopoly in G. Recall the definition, $U = V \setminus M$.

For a vertex $v \in V$, define

$$surplus(v) \quad = \quad |N_{G_1}(v) \cap M| - |N_{G_1}(v) \cap U|, \tag{19}$$

and let $deficit(v) = -surplus(v)$. Note that the definitions $deficit(v)$ and $surplus(v)$ are different from those in the previous sections. By definition, v is controlled by M in G_1 if and only if $surplus(v) \geq 0$ (i.e., $deficit(v) \leq 0$). Let M_-, M_0 and M_+ be the sets of vertices $v \in M$ such that $surplus(v) < 0$, $surplus(v) = 0$ and $surplus(v) > 0$, respectively, and let U_-, U_0 and U_+ be the sets of vertices $v \notin M$ such that $surplus(v) < 0$, $surplus(v) = 0$ and $surplus(v) > 0$, respectively. By definition, $M = M_- \cup M_0 \cup M_+$ and $U = U_- \cup U_0 \cup U_+$.

Let $G = (V, E)$ be a solution to the min-neighborhood monopoly problem, and let $E = E_1 \cup \Delta$, where $\Delta \subseteq D$. Since minimizing E is equivalent to minimizing Δ, we discuss properties of Δ instead of those of E. Based on discussions in Section 2, without loss of generality, we can assume

$$\Delta \quad \subseteq \quad D(M, M) \cup D(U_-, M). \tag{20}$$

We first prove the following lemma:

Lemma 5 *For each $v \in U_-$, we have*

$$|\Delta(v, M)| \quad = \quad \mathit{deficit}(v). \tag{21}$$

Proof. First observe that $|\Delta(v, M)| \geq \mathit{deficit}(v)$, since otherwise,

$$\begin{aligned} |N_G(v) \cap M| - |N_G(v) \cap U| &= (|N_{G_1}(v) \cap M| + |\Delta(v, M)|) - |N_{G_1}(v) \cap U| \\ &= |\Delta(v, M)| - \mathit{deficit}(v) \ < \ 0, \end{aligned}$$

a contradiction, since this would mean that v was not controlled by M in G.

Next suppose that $|\Delta(v, M)| > \mathit{deficit}(v)$. Let $\Delta' = \Delta - \{(v, w)\}$ for some $(v, w) \in \Delta(v, M)$, and let $G' = (V, E')$, where $E' = E_1 \cup \Delta'$). By (20), we have

$$\begin{aligned} |N_{G'}(v) \cap M| - |N_{G'}(v) \cap U| &\geq 0, \\ |N_{G'}(w) \cap M| - |N_{G'}(w) \cap U| &= |N_G(w) \cap M| - |N_G(w) \cap U| + 1 \ \geq \ 0, \\ |N_{G'}(u) \cap M| - |N_{G'}(u) \cap U| &= |N_G(u) \cap M| - |N_G(u) \cap U| \ \geq \ 0 \quad \text{for all} \quad u \neq v, w. \end{aligned}$$

It follows from these inequalities that M remains to be a monopoly in G'. Since $|E'| < |E|$, this contradicts the assumption on E. Thus $|\Delta(v, M)| = \mathit{deficit}(v)$ for all $v \in U_-$. □

The following corollary is a direct consequence of the above lemma.

Corollary 2

$$|\Delta(U_-, M)| \quad = \quad \sum_{v \in U_-} \mathit{deficit}(v).$$

□

Corollary 2 implies that for every minimum Δ, $\Delta(U_-, M)$ is of the same size, and hence Δ contains a minimum $\Delta(M, M)$.

We now associate a graph $G^* = (V^*, E^*)$ with $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ and $M \subseteq V$ as follows:

$$\begin{aligned} V^* &= V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5, \\ E^* &= \bigcup_{v \in U_-, w \in M} E_{(v,w)} \cup \bigcup_{v \in M_-} E_v \cup E_a, \end{aligned}$$

where

$$\begin{aligned} V_1 &= \{v_1, v_2, \ldots, v_{\mathit{deficit}(v)} \,|\, v \in M_-\}, \\ V_2 &= \{v_1, v_2, \ldots, v_{\mathit{deficit}(v)} \,|\, v \in U_-\}, \end{aligned}$$

$$
\begin{aligned}
V_3 &= \{v_1, v_2, \ldots, v_{surplus(v)} \,|\, v \in M_+\}, \\
V_4 &= \{x_{(v,w)}, y_{(v,w)} \,|\, (v,w) \in D(U_-, M)\}, \\
V_5 &= \{z_{<v,w>}, z_{<w,v>} \,|\, (v,w) \in D(M,M)\}, \\
E_{(v,w)} &= \{(v_i, x_{(v,w)}), (x_{(v,w)}, y_{(v,w)}), (y_{(v,w)}, z_{<w,u>}), (y_{(v,w)}, w_j) \,|\, v_i \in V_2, w_j \in V_3, z_{<w,u>} \in V_5\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ (v,w) \in D(U_-, M_+), \\
E_{(v,w)} &= \{(v_i, x_{(v,w)}), (x_{(v,w)}, y_{(v,w)}), (y_{(v,w)}, z_{<w,u>}) \,|\, v_i \in V_2, z_{<w,u>} \in V_5\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \ (v,w) \in D(U_-, M_0 \cup M_-), \\
E_v &= \{(v_i, z_{<v,w>}) \,|\, v_i \in V_1, (v,w) \in D(v,M)\} \quad \text{for } \ v \in M_-, \\
E_a &= \{(z_{<v,w>}, z_{<w,v>}) \,|\, z_{<v,w>}, z_{<w,v>} \in V_5\}.
\end{aligned}
$$

Here we assume that $s_{(v,w)} = s_{(w,v)}$ for $s = x, y$, and $z_{<v,w>} \neq z_{<w,v>}$. For a vertex $v \in U_-$, let $E_v = \bigcup_{w \in M} E_{(v,w)}$. Then E^* can be written as

$$
E^* \;=\; \bigcup_{v \in U_- \cup M_-} E_v \cup E_a. \tag{22}
$$

We define a function, *weight* : $E^* \mapsto \mathbb{R}^+$ by

$$
weight(e) \;=\; \begin{cases} 1 & \text{if } e \in E_a, \\ 3L & \text{if } e \in E_b, \\ 2L & \text{otherwise,} \end{cases}
$$

where $E_b = \{(x_{(v,w)}, y_{(v,w)}) \,|(v,w) \in D(U_-, M)$ and L is a number greater than $|E_a|$.

For example, Figure 6 shows the graph G^* associated with the problem instance given in Figure 1.

In the following, we show that computing a maximum weighted matching in G^* is polynomially equivalent to the min-neighborhood monopoly problem. Let S be a maximum weighted matching in G^*, and let *Weight* denote its weight; i.e., $Weight = weight(S)$, where $weight(T) = \sum_{e \in T} weight(e)$ for a set $T \subseteq E^*$.

Lemma 6 *Let Weight and L be as defined above. Then Weight* $\geq \Theta$*, where*

$$
\Theta \;=\; 3L|D(U_-, M)| + L \sum_{v \in U_-} deficit(v) + 2L \sum_{v \in M_-} deficit(v). \tag{23}
$$

Proof. Given a smallest graph $G = (V, E = E_1 \cup \Delta)$ such that $\Delta \subseteq D$ and a monopoly M in G, we shall construct a matching T in G^* such that $weight(T) = \Theta$, which will complete the proof.

To construct such a T, we first introduce a mapping α_v for each $v \in M$. For a vertex $v \in M_- \cup M_0$, we have

$$
|\Delta(v, M)| \;\geq\; |\Delta(v, U)| + deficit(v).
$$

Let α_v, $v \in M_- \cup M_0$, be an arbitrary injective mapping from $\Delta(v, U) \cup \{1, 2, \ldots, deficit(v)\}$ to $\Delta(v, M)$, where we assume that $\Delta(v, U) \cap \{1, 2, \ldots, deficit(v)\} = \emptyset$. Similarly, for a vertex $v \in M_+$, we have

$$
|\Delta(v, M)| + surplus(v) \;\geq\; |\Delta(v, U)|.
$$

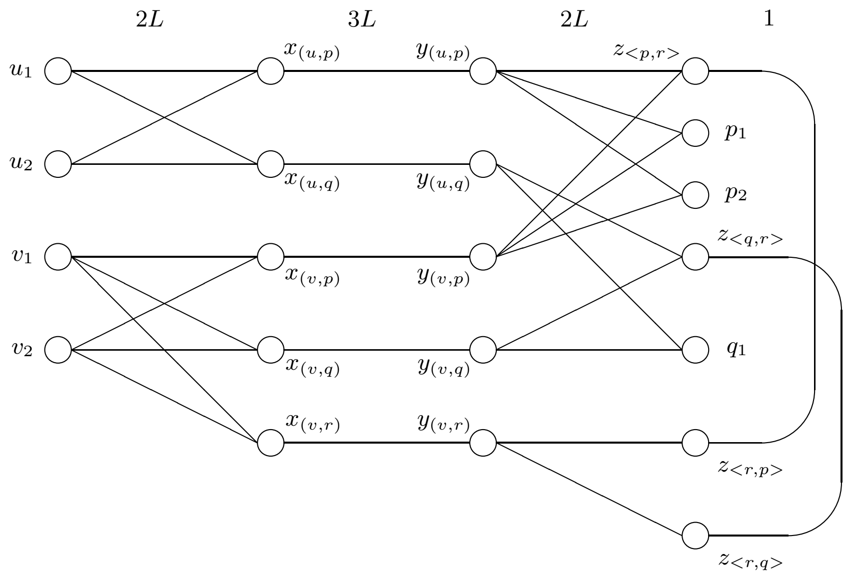

Figure 6: The graph $G^* = (V^*, E^*)$ and $weight : E^* \mapsto \mathbb{R}^+$ associated with G_1, G_2 and M in Figure 1.

Let α_v, $v \in M_+$, be an arbitrary injective mapping from $\Delta(v, U)$ to $\Delta(v, M) \cup \{1, 2, \ldots, surplus(v)\}$, where we assume that $\Delta(v, M) \cap \{1, 2, \ldots, surplus(v)\} = \emptyset$.

Intuitively, α_v shows how to make up for the "expenditure" $|\Delta(v, U)| + deficit(v)$ (resp., $|\Delta(v, U)|$) by its "income" $|\Delta(v, M)|$ (resp., $|\Delta(v, M)| + surplus(v)$).

Now we define a matching T in G^*. For a vertex $v \in M_-$, define a set T_v of edges in G^* by

$$T_v \quad = \quad \{(v_i, z_{<v,w>}) \,|\, i = 1, 2, \ldots, deficit(v), \alpha_v(i) = (v, w)\}. \tag{24}$$

For a vertex $v \in U_-$, let $\Delta(v, M) = \{e_1, e_2, \ldots, e_{deficit(v)}\}$ (Note that $|\Delta(v, M)| = deficit(v)$ by Lemma 5). Moreover, for each $e_i = (v, w) \in \Delta(v, M)$, let

$$\beta(e_i) \quad = \quad \begin{cases} z_{<w,u>} & \text{if } \alpha_w(e_i) = (w, u) \in \Delta(w, M), \\ w_k & \text{if } \alpha_w(e_i) = k \in \{1, 2, \ldots, surplus(w)\}. \end{cases}$$

For a vertex $v \in U_-$, define a set T_v of edges in G^* by

$$T_v \quad = \quad \{(v_i, x_{e_i}), (y_{e_i}, \beta(e_i)) \,|\, i = 1, 2, \ldots, deficit(v)\} \cup \{(x_e, y_e) \,|\, e \in D(v, M) \setminus \Delta(v, M)\}. \tag{25}$$

Let $T = \bigcup_{v \in M_- \cup U_-} T_v$. Since every α_v is injective, T forms a matching in G^*. We have $weight(T_v) = \Theta_v$, where

$$\Theta_v \quad = \quad \begin{cases} 2L \times deficit(v) & \text{if } v \in M_-, \\ 3L\,|D(v, M)| + L \times deficit(v) & \text{if } v \in U_-. \end{cases} \tag{26}$$

This completes the proof. □

Let T be a matching in G^*. For a vertex $v \in U_- \cup M_-$, let $T_v = T \cap E_v$. Then T can be written as

$$T \quad = \quad \bigcup_{v \in U_- \cup M_-} T_v \;\cup\; T_a,$$

where $T_a = T \cap E_a$. Note that $T_v \cap T_w = \emptyset$ for each pair, $v, w \in U_- \cup M_-$, and $T_v \cap T_a = \emptyset$ for each $v \in U_- \cup M_-$.

Lemma 7 *Let T be a matching in G^* such that weight$(T) \geq \Theta$. Then we have weight$(T_v) = \Theta_v$ holds, where Θ_v is given by* (26)*.*

Proof. Let us first consider T_v for $v \in U_-$. Note that $E_v = \bigcup_{(v,w)\in D(v,M)} E_{(v,w)}$. It is clear that every $E_{(v,w)}$ forms a tree, and that the size of a maximum (resp., second and third largest) matching in the sugraph $(V, E_{(v,w)})$ is $4L$ (resp., $3L$ and $2L$). Note that at most *deficit*(v) vertices, w, satisfy *weight*$(T_v \cap E_{(v,w)}) = 4L$. Thus,

$$\mathit{weight}(T_v) \quad \leq \quad \Theta_v. \tag{27}$$

Moreover, if *weight*$(T_v) < \Theta_v$, then

$$\mathit{weight}(T_v) \quad \leq \quad \Theta_v - L. \tag{28}$$

As for a vertex $v \in M_-$, E_v forms a "star" having *deficit*(v) leaves. Since all edges in E_v have the same weight $2L$, (27) must hold. Also *weight*$(T_v) < \Theta_v$ implies (28).

Finally, since *weight*$(T_a) \leq$ *weight*$(E_a) < L$, if some $v \in U_- \cup M_-$ satisfies *weight*$(T_v) < \Theta_v$, (27) and (28) imply

$$\mathit{weight}(T) \quad = \quad \sum_{v\in U_-\cup M_-} \mathit{weight}(T_v) \ + \ \mathit{weight}(T_a) \ < \ \Theta - L + L \ = \Theta,$$

which contradicts the assumption on T. Therefore, each vertex $v \in U_- \cup M_-$ satisfy *weight*$(T_v) = \Theta_v$. □

The proof of Lemma 7 also implies *Weight* $< \Theta + L$, and it follows that every matching T such that *weight*$(T) \geq \Theta$ produces a desirable Δ (in the sense that M is a monopoly in $G = (V, E_1 \cup \Delta)$), by reversing the construction in the proof of Lemma 6 to T.

We shall now show that every maximum weighted matching S corresponds to a minimum added edge set Δ in G_1.

Lemma 8 *Let Weight, L and Θ be as defined above, and let Δ be a minimum edge set added to G_1 so that M is a monopoly in $G = (V, E_1 \cup \Delta)$. Then we have*

$$\mathit{Weight} - \Theta \quad = \quad |D(M, M)| - |\Delta(M, M)|.$$

Proof. For a given Δ, let us construct a matching $T = \bigcup_{v\in W_-\cup U_-} T_v$ in G^*, where T_v is defined either by (24) or by (25). Note that E_a is a matching in G^*. Let T_a be the set of edges $(z_{<v,w>}, z_{<w,v>})$ in E_a such that neither $z_{<v,w>}$ nor $z_{<w,v>}$ appears in T. Then $S = T \cup T_a$ is clearly a matching in G^*. We have

$$\mathit{weight}(S) - \Theta \quad = \quad \mathit{weight}(T_a) \ = \ |T_a|. \tag{29}$$

From the construction of T, $(z_{<v,w>}, z_{<w,v>}) \in T_a$ if and only if $(v, w) \notin \Delta$, and hence $(v, w) \in D(M, M) - \Delta(M, M)$ (see the proof of Lemma 6). It follows from $D(M, M) \supseteq \Delta(M, M)$ and (29) that *weight*$(S) - \Theta = |D(M, M)| - |\Delta(M, M)|$. Therefore, we have

$$\mathit{Weight} - \Theta \quad \geq \quad \mathit{weight}(S) - \Theta \ = \ |D(M, M)| - |\Delta(M, M)|. \tag{30}$$

On the other hand, let T be a maximum weighted matching in G^*. Then by Lemma 6, we have $weight(T) \geq \Theta$. Since $weight(T_v) = \Theta_v$ holds for every $v \in U_- \cup M_-$ by Lemma 7, we have

$$Weight - \Theta \quad = \quad weight(T_a) \; = \; |T_a|.$$

It follows from the discussion before this lemma that we can construct a desirable Δ' (which may not be minimum) from this T. Similarly to the above case, we have $(z_{<v,w>}, z_{<w,v>}) \in T_a$ if and only if $(v, w) \notin \Delta'$ (and hence $(v, w) \in D(M, M) - \Delta'(M, M)$). Moreover, it follows from Corollary 2 that Δ is minimum if and only if so is $\Delta(M, M)$. Thus

$$|D(M,M)| - |\Delta(M,M)| \quad \geq \quad |D(M,M)| - |\Delta'(M,M)| \quad = \quad Weight - \Theta. \tag{31}$$

This, together with (30), proves the lemma. □

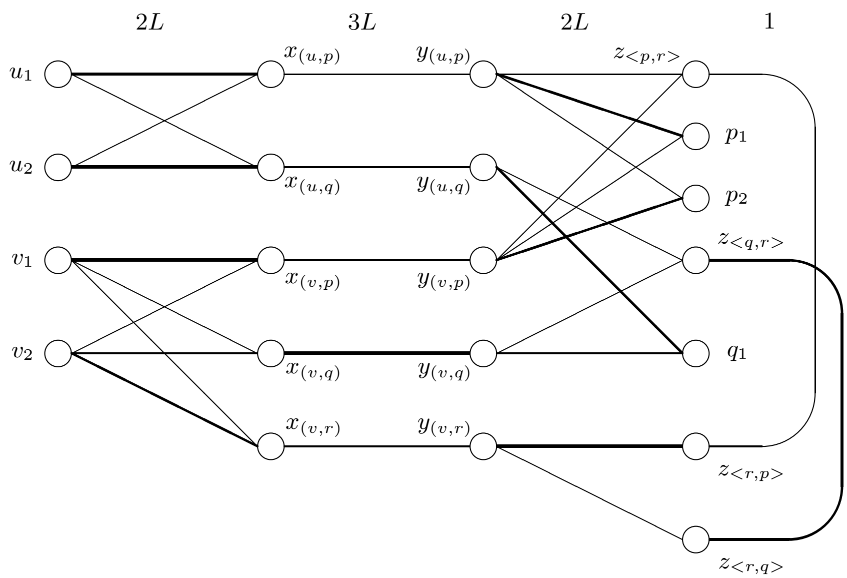

Figure 7: A maximum weighted matching in $G^* = (V^*, E^*)$ which was given in Figure 6.

For example, Figure 7 shows a maximum weighted matching in G^* given in Figure 6. In fact, this matching corresponds to a desired graph G given in Figure 8.

Let us note that for the graph G^* we have $|V^*| = O(m_2)$, $|E^*| = O(m_2^2)$ and $\max weight(e^*) = O(m_2)$, where $m_2 = |E_2|$. Since a maximum weighted matching in such a graph can be computed in $\tilde{O}(m_2^{5/2})$ time [4], we have the following theorem:

Theorem 3 *The min-neighborhood monopoly problem can be solved in $\tilde{O}(m_2^{5/2})$ time.* □

5 Max Controlled Set Problem

Unfortunately, this problem is intractable, even if we restrict ourselves to the edge-augmentation and the edge-deletion problems.

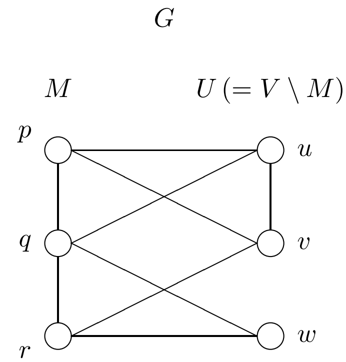

Figure 8: The graph G corresponding to a maximum weighted matching in Figure 7.

Theorem 4 *The max controlled set problem is NP-hard, even if G_1 is the empty graph.*

Proof. We transform the following well-known NP-hard problem, INDEPENDENT SET, to this problem.

Problem: INDEPENDENT SET [5]
Input: An undirected graph $G^* = (V^*, E^*)$, where $V^* = \{1, 2, \ldots, n\}$ and $|E^*| = m$.
Output: An independence number $\alpha(G^*)$, i.e., the size of a maximum set $Y \subseteq V^*$ such that no two vertices in Y are joined by an edge in E^*.

For this G^*, we define an undirected graph $G_2 = (V, E_2)$ by

$$\begin{aligned} V &= V^* \cup V_1 \cup V_2, \\ E_2 &= E_a \cup E_b, \end{aligned}$$

where

$$\begin{aligned} V_1 &= \{n+1, n+2, \ldots, 2n\}, \\ V_2 &= \{x_e \,|\, e \in E\}, \\ E_a &= \{(i, x_e) \,|\, i \in V^*, x_e \in V_2, i \in e\}, \\ E_b &= \{(i, n+i) \,|\, i \in V^*\}. \end{aligned}$$

Let $G_1 = (V, \emptyset)$ and $M = V^*$ $(U = V \setminus V^* (= V_1 \cup V_2))$, and define

$$\mathcal{W} = \{W \subseteq V \,|\, W \text{ is controlled by } M \text{ in some subgraph of } G_2\}. \tag{32}$$

We claim that

$$\max_{W \in \mathcal{W}} |W| = \alpha(G^*) + n + m,$$

which will complete the proof.

Let us first show $\max_{W \in \mathcal{W}} |W| \geq \alpha(G^*) + n + m$. Let $Y \subseteq V^*$ be a maximum independent set in G^*, i.e., $|Y| = \alpha(G^*)$. Define a graph $G = (V, E)$ by

$$E \quad = \quad E_b \cup \{(i, x_e) \in E_a \,|\, i \in V^* \setminus Y\}.$$

Since $G = 1$ is the empty graph, we have $E_1 = \emptyset$. Since $|N_G(i) \cap M| - |N_G(i) \cap U| = 0$ for each $i \in V_1 \cup Y$, $V_1 \cup Y$ is controlled by M in G. Moreover, since $V^* \setminus Y$ forms a vertex cover in G^*, i.e., it contains either v or w for each edge $(v, w) \in E^*$, $|N_G(x_e) \cap M| - |N_G(x_e) \cap U| \geq 0$ holds for every $x_e \in V_2$, and thus V_2 is also controlled by M in G. In total, we have

$$\max_{W \in \mathcal{W}} |W| \quad \geq \quad |Y| + |V_1| + |V_2| \quad = \quad \alpha(G^*) + m + n. \tag{33}$$

To show the opposite inequality, let $G = (V, E)$ be a subgraph of G_2 in which M controls as many vertices in V as possible (Note that G is a supergraph of $G_1 = (V, \emptyset)$), and let W^* be the set controlled by M in G. For each $v \in U \setminus W^*$, we modify the edge set E to $E \cup \{(v, w)\}$ for some $(v, w) \in E_2 \setminus E$. Let $G' = (V, E')$ be the resulting graph, and let W' be the set controlled by M in G'. Clearly G' is a subgraph of G_2, and moreover, we have $|W'| \geq |W^*|$, since the modification of E to $E \cup \{(v, w)\}$ makes v controlled by M and does not affect vertices except for v and w. We therefore assume that $U \subseteq W^*$ without loss of generality. Now define $Y \subseteq V^*$ by

$$Y \quad = \quad \{i \in V^* \,|\, \text{there exists no } x_e \in V_2 \text{ such that } (i, x_e) \in E\}.$$

Since $V_2 \subseteq W^*$, $V^* \setminus Y$ forms a vertex cover in G^* (and hence Y is independent). We have

$$|W^*| \; (= \max_{W \in \mathcal{W}} |W|) \quad = \quad |Y| + |V_1| + |V_2| \quad \leq \quad \alpha(G^*) + n + m,$$

which, together with (33), proves our claim. □

Theorem 5 *The max controlled set problem is NP-hard, even if G_2 is a complete graph.*

Proof. We transform INDEPENDENT SET to this problem (see the proof of Theorem 4). We assume without loss of generality that $m > n$ and G^* is cubic,[2] since this restriction does not affect the NP-hardness of INDEPENDENT SET [5]. For this G^*, we define an undirected graph $G_1 = (V, E_1)$ by

$$\begin{aligned} V &= V^* \cup V_1 \cup V_2, \\ E_1 &= E_a \cup E_b \cup E_c \cup E_d, \end{aligned}$$

where

$$\begin{aligned} V_1 &= \{n+1, n+2, \ldots, m+1\}, \\ V_2 &= \{x_e \,|\, e \in E\}, \\ E_a &= \{(x_e, x_{e'}) \,|\, x_e, x_{e'} \in V_2\}, \\ E_b &= \{(i, x_e) \,|\, i \in V^*, x_e \in V_2, i \notin e\}, \\ E_c &= \{(i, x_e) \,|\, i \in V_1, x_e \in V_2\}, \\ E_d &= \{(i, j) \,|\, i, j \in V^* \cup (V_1 \setminus \{m+1\})\}. \end{aligned}$$

[2] A graph is called *cubic* if every vertex has degree 3.

Let G_2 be a complete graph over the vertex set V. and $M = V_2\,(\subseteq V)$.

Let $U = V \setminus M\,(= V^* \cup V_1)$, and first consider the surplus of $v \in V$, where

$$surplus(v) \;=\; |N_{G_1}(v) \cap M| - |N_{G_1}(v) \cap U|. \qquad \text{(This corresponds to (19).)}$$

For each $x_e \in V_2$, we have

$$\begin{aligned} surplus(x_e) \;&=\; (|E_a(x_e)| + 1) - (|E_b(x_e)| + |E_c(x_e)|) \\ &=\; m - ((n-2) + (m+1-n)) \;=\; 1, \end{aligned} \tag{34}$$

where $E_\alpha(v) = \{(v, w) \in E_\alpha\}$, and for each $i \in V_1 \setminus \{m+1\}$ we have

$$surplus(i) \;=\; |E_c(i)| - (|E_d(i)| + 1) \;\geq\; m - ((m-1)+1) \;=\; 0, \tag{35}$$

and

$$surplus(m+1) = m - 1 > 0.$$

Finally, since G^* is cubic, for each $i \in V^*$ we have

$$surplus(i) \;=\; |E_b(i)| - (|E_d(i)| + 1) \;=\; (m-3) - ((m-1)+1) \;=\; -3. \tag{36}$$

Let

$$\mathcal{W} \;=\; \{W \subseteq V \mid W \text{ is controlled by } M \text{ in some supergraph of } G_1\}.$$

We now claim that

$$\max_{W \in \mathcal{W}} |W| \;=\; \alpha(G^*) + 2m - n + 1,$$

where $\mathcal{W}$ is given in (32). This completes the proof.

Let us first show $\max_{W \in \mathcal{W}} |W| \geq \alpha(G^*) + 2m - n + 1$. Let $Y \subseteq V^*$ be a maximum independent set in G^*, i.e., $|Y| = \alpha(G^*)$. Let

$$E \;=\; E_1 \cup \Delta, \text{ where } \Delta = \{(i, x_e) \in D \mid i \in Y\}.$$

Since $|D(i, V_2)| = 3$ for all $i \in Y$, (36) implies that every vertex $i \in Y$ is controlled by M in $G = (V, E)$. Moreover, since Y is an independent set in G, (34) implies that all vertices in V_2 are still controlled by M in G. Finally, since $D(V_1, V) \cap \Delta = \emptyset$, all vertices in V_1 are controlled by M in G. Thus we have

$$\max_{W \in \mathcal{W}} |W| \;\geq\; |Y| + |V_2| + |V_1| \;=\; \alpha(G^*) + 2m - n + 1. \tag{37}$$

Let us next show the opposite inequality. Consider a maximum controled set W^* having the maximum $|W^* \cap V_2|$ (i.e., $W^* \in \max \mathcal{W}$ such that $|W^* \cap V_2| = \max_{W \in \max \mathcal{W}} |W \cap V_2|$, where $\max \mathcal{W} = \{W' \in \mathcal{W} \mid |W'| = \max_{W \in \mathcal{W}} |W|\}$). We assume that this W^* is controlled by M in $G = (V, E)$, where $E = E_1 \cup \Delta$. Similarly to the discussion in Section 2, without loss of generality, we can assume that $\Delta \subseteq D(V, V_2)$. This implies $W^* \supseteq V_1$. Moreover, we have $W^* \supseteq V_2$. Assuming to the contrary that there is a vertex $x_e \in V_2 \setminus W^*$. By (34) and $|D(x_e)| = 2$, Δ contains two edges (i, x_e) and (j, x_e), where $e = (i, j)$. Now, let G' be the graph obtained from G by removing (i, x_e), Clearly this G' is a supergraph of G_1. Moreover,

$W' = (W \setminus \{i\}) \cup \{x_e\}$ is controlled by M in G'. By the maximality of W^*, W' is also a maximum controllable set. This contradicts the assumption on W^*, since $|W^{**} \cap V_2| > |W^* \cap V_2|$. We therefore have $W^* \supseteq V_1 \cup V_2$.

Finally, since $W^* \supseteq V_2$, $W^* \cap V^*$ is an independent set in G^*. Hence

$$|W^*| \ (= \max_{W \in \mathcal{W}} |W|) \quad = \quad |W^* \cap V^*| + |V_2| + |V_1| \ \leq \ \alpha(G^*) + 2m - n + 1,$$

which, together with (37), proves our claim. □

Since the max controlled set problem seems to be intractable, we consider an approximation algorithm. We now present a simple approximation algorithm which guarantees an approximation ratio of 2.

For two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, and a set $M \subseteq V$, we construct two graphs $G^+ = (V, E^+)$ and $G^{++} = (V, E^{++})$ for G_1 and G_2 by

$$\begin{aligned} E^+ &= E_1 \cup D(M, M), \text{ and} \\ E^{++} &= E_1 \cup D(M, M) \cup D(U, M), \end{aligned}$$

respectively, where $U = V \setminus M$. Let W^+ and W^{++} be the sets of vertices in V which are controlled by M in G^+ and G^{++}, respectively. The following lemma is immediate from the definitions of G^+ and G^{++}.

Lemma 9 *Let W^+ and W^{++} be as defined above. Let $\mathcal{W}$ be a family of sets $W \subseteq V$ which are controlled by M in some graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$. Then we have*

$$\begin{aligned} |W^+ \cap M| &= \max_{W \in \mathcal{W}} |W \cap M|, \\ |W^{++} \cap U| &= \max_{W \in \mathcal{W}} |W \cap U|. \end{aligned}$$

Lemma 10 *Let W^+ and W^{++} be as defined above. Let W^* be the larger of the two, i.e., $|W^*| = \max\{|W^+|, |W^{++}|\}$. Then W^* satisfies*

$$|W^*| \quad \geq \quad 1/2 \max_{W \in \mathcal{W}} |W|.$$

Proof. It follows from Lemma 9 that $\max_{W \in \mathcal{W}} |W| \leq |W^+| + |W^{++}| \leq 2W^*$. □

Theorem 6 *Given two graphs $G_1 = (V, E_1)$, $G_2 = (V, E_2)$, and a set $M \subseteq V$, we can compute in polynomial time a graph $G = (V, E)$ with $E_1 \subseteq E \subseteq E_2$ such that the size of the set controlled by M in G is at least half of that of a maximum controlled set.*

6 Conclusion

This paper discussed edge augmentation and deletion problems when the number of vertices controlled by a given set M of vertices is held at maximum. These problems were shown to be NP-complete in general, by a transformation from the maximum independent set problem. However, it can be determined in polynomial time if the addition (or deletion) of a set of edges can make M control all vertices, by reducing it to the network flow problem.

One can easily extend the positive results in the following way. For a function f on V, a vertex $v \in V$ is said to be *f-controlled* if $|N_G(v) \cap M| - |N_G(v) \setminus M| \geq f(v)$. Then the problems corresponding to the monopoly verification, max-neighborhood monopoly and min-neighborhood monopoly problems can be solved in polynomial time by applying the network flow and matching arguments to them, respectively, and the approximation argument also carries over to the max f-controlled set problem. However, the NP-hardness result does not hold for every function f. For example, if $f(v) = |V|$ for all $v \in V$, then the max f-controlled set problem is polynomially solvable.

Some problems remain to be addressed in future work. One issue is the search for faster or simpler algorithms for our problems. Another issue is to consider max controlled set problem for special classes of graphs.

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