

Simple Characterization of Polygons Searchable by 1-Searcher

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Abstract

Suppose intruders are in a dark polygonal room and they can move arbitrarily fast, trying to avoid detection. A *boundary 1-searcher* moves along the polygon boundary, equipped with a flash light that she can direct in any direction. A polygon is *searchable* if there is a schedule for the searcher in order to detect the intruders no matter how they move. We identify three simple *forbidden patterns* such that a given polygon is searchable by a boundary 1-searcher if and only if it has none of them. The concept of *sweeping* the visibility diagram greatly facilitates the proof.¹

1 Introduction

Suzuki and Yamashita defined a *k-searcher* equipped with *k* flashlights who tries to detect intruders who can move faster than the scanning speed of flashlights by eventually illuminating them [8]. For recent results on this topic, the reader is referred to [1, 2, 9]. Two points *a, b* in a simple polygon *P* are said to be *mutually visible* if $\overline{ab} \subseteq P$, where ∂P denotes the boundary of *P*. In [5] Park et al. worked on the characterization of polygons that are searchable by a general 1-searcher. Tan characterizes the polygons that are searchable by a *boundary 1-searcher* [9]. Unfortunately, the proofs of correctness in these papers are several pages long and difficult to follow.

The aim of this paper is to characterize the simple polygons that are searchable by a boundary 1-searcher, in terms of a new set of three simple forbidden patterns. One of them is a special case of the pattern given in [8]. The other two resemble those in [9]. Most importantly, we use a high-level tool, called the *V-diagram* developed in [2], to make the proof of correctness significantly shorter (about 3/4 of a page), in contrast to several pages for existing proofs.

2 Preliminaries

2.1 Notation

A (simple) polygon *P* consists of n (≥ 3) vertices and n edges connecting adjacent vertices. The vertices preceding and succeeding *v* in the clockwise order are denoted by *Pred*(*v*) and *Succ*(*v*), respectively. A *reflex vertex* is one whose interior angle is larger than 180° . For a reflex vertex *v*, let *F*(*v*) (resp. *B*(*v*)) denote the point on ∂P where the extension of the edge (*Pred*(*v*), *v*) (resp. (*Succ*(*v*), *v*)) towards the interior of *P* hits ∂P for the first time. Let $a, b \in \partial P$. The clockwise section of ∂P from point *a* to point *b* is denoted by $\partial P[a, b]$. If $c \in \partial P[a, b] \setminus \{a, b\}$, we write $a \prec c \prec b$. For a reflex vertex *v*, the part of *P* enclosed by $\partial P[v, B(v)]$ (resp. $\partial P[F(v), v]$) and the line segment $vB(v)$ (resp. $vF(v)$) is called the *clockwise* (*cw*) (resp. *counterclockwise* (*ccw*)) *component* associated with *v*.

We fix an arbitrary point on ∂P as the origin, and measure all distances along ∂P clockwise from the origin. Let $|\partial P|$ denote the length of ∂P . For $x \in \mathbf{R}$ (the set of all real numbers), *x* represents the point on ∂P that is at distance $x - k|\partial P|$ from the origin, where *k* is an integer such that $0 \leq x - k|\partial P| < |\partial P|$. We thus can consider *x* also as a point on ∂P . Note that there are infinitely many real numbers $x \in \mathbf{R}$ that represent a single point on ∂P .

2.2 Visibility diagram and search path

Let $x, y \in \mathbf{R}$. The *visibility space*, denoted by \mathcal{V} , consists of the infinite area between and including the lines $y = x$ (*start line* \mathcal{S}) and $y = x - |\partial P|$ (*goal line* \mathcal{G}), as shown in Fig. 1 [2].

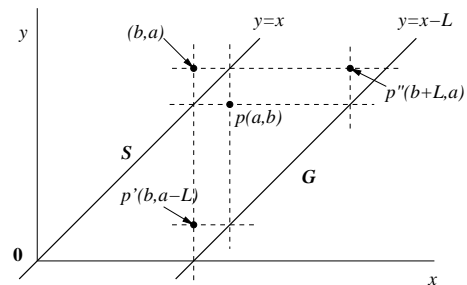


Figure 1: Visibility space ($L = |\partial P|$).

The *visibility diagram* (*V-diagram* for short) for a given polygon is drawn in \mathcal{V} by shading some areas in it

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¹This write-up is a slightly revised version of [3].

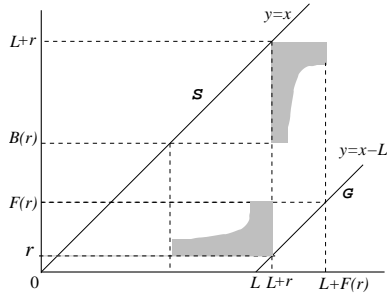


Figure 2: SE and NW barriers due to reflex vertex r ($L = |\partial P|$).

gray as follows: point $(x, y) \in \mathcal{V}$ is gray if points x and y are not mutually visible. We use x (y) to represent the beam head (searcher) position. Note that in Fig. 1 point p is gray if and only if points p' and p'' are also gray. Each reflex vertex r gives rise to two shaded areas in each section of length $|\partial P|$ in the V-diagram, as shown in an idealized form in Fig. 2.

We call each such shape a *barrier* and a barrier whose corner is touching line \mathcal{S} or \mathcal{G} is called a *southeast (SE)* or *northwest (NW) barrier*, respectively [2]. See Fig. 3 for an example of a polygon and its V-diagram. (Imagine that the blackened area is gray.) A *skeleton V-diagram* if obtained from a V-diagram by replacing each SE (NW) barrier by its *backbones*, consisting of a horizontal and vertical line segments that form its upper (lower) and left (right) faces, respectively [2].

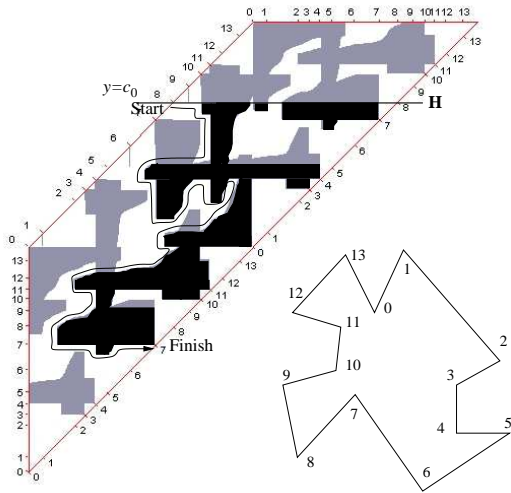


Figure 3: An example polygon and its V-diagram.

A *search path* is a path from \mathcal{S} to \mathcal{G} that stays within white areas, except that it may cross gray areas from right to left, e.g., the curve from Start to Finish in Fig. 3. Crossing a gray area from right to left reflects the movement (a jump) of the beam head from the cur-

rent position to another boundary point as the beam is moved to the left [2, 4].

Theorem 1 [2] *A given polygon is searchable by a boundary 1-searcher if and only if there is a search path in its V-diagram.* \square

3 Searchability test by sweeping V-diagram

It is known that a polygon is searchable by a boundary one-way ∞ -searcher (who has 360° vision) if and only if it is searchable by a boundary 1-searcher [7]. In terms of Fig. 3, let us see how an ∞ -searcher can clear a polygon. Suppose she is initially positioned at an arbitrary point $c_0 \in \partial P$, where $c_0 \in \mathbf{R}$. In the V-diagram, let H denote the intersection of line $y = c_0$ and the gray region. Then H corresponds to the parts of ∂P that the ∞ -searcher cannot see from point c_0 .

The counter-clockwise movement of the ∞ -searcher can be simulated by the downward movement of the line $y = c$, as c is decreased from c_0 . Let X be a maximal contiguous segment of H . Such an X may grow or shrink as c is decreased. If X shrinks to the empty set, it implies that part of the boundary has been cleared by the ∞ -searcher. Suppose a new section is added to H as c is decreased. This is due to the fact that a previously clear section has become invisible because it became hidden behind some reflex corner. Therefore, this new section is still clear, although it is currently invisible.

Sweeping Rule: Draw a horizontal line $y = c_0$ across the V-diagram at an arbitrary position. If it doesn't cross the gray region, stop. Otherwise, let H denote the intersection between $y = c_0$ and the gray region. Paint each point p in the gray region black, if p can be reached from a point in H by a path that moves left, right or downward without leaving the gray region. \square

To visualize the sweeping operation, imagine that H emits black fluid that flows downward using the gray region of the V-diagram as a conduit. The fluid flows left, right or downwards (but not upwards) constrained by the conduit, making the parts of the conduit it comes in contact black.

Suppose that the painting by the sweeping operation terminates as in Fig. 3. We show later that whether it terminates or not is independent of the choice of c_0 . A search path can be constructed by following the boundary of the painted areas, starting just below line $y = c_0$ at the start line S . See the path from Start to Finish shown in Fig. 3. The line $y = c_0$ is considered black. It follows from results in Sec. 5 of [7] that

Theorem 2 *Polygon P is searchable by a boundary 1-searcher if and only if the area of its V-diagram that is painted black by the sweeping operation is finite.* \square

4 Characterization

We first identify some patterns to be used to define our three forbidden patterns. Two reflex vertices u and v such that neither of them is in the cw component of the other (i.e., $u \prec B(u) \prec v$ and $v \prec B(v) \prec u$ hold) form a *cw exclusion pair* (EXP_{cw}, for short). For example, see u and v in Fig. 4(a). Similarly, two reflex vertices u and v such that neither of them is in the ccw component of the other (i.e., $u \prec F(v) \prec v$ and $v \prec F(u) \prec u$ hold) form a *ccw exclusion pair* (EXP_{ccw}, for short). For example, see u and v in Fig. 4(b). Finally, two reflex vertices u and v such that u (v) is not in the ccw (cw) component of v (u) (i.e., $u \prec \{B(u), F(v)\} \prec v$ holds) form a *symmetric exclusion pair* (EXPS_{sym}, for short), also called a *trap*, *deadlock*, or *BF-pair* [9]. In this case, each point in $\partial P(v, u)$ is said to be *trapped*. For example, see u and v in Fig. 4(c). Vertices u and v need not be mutually visible in Fig. 4. Refer to Fig. 5(c) to see where trap $\{u, v\}$ is located in the V-diagram. Note that the backbones of an exclusion pair intersect, as seen in the skeleton V-diagrams in Fig. 5.

An *EXP_{cw} couple* (2EXP_{cw} for short) consists of three reflex vertices, u, v , and w such that both $\{u, v\}$ and $\{u, w\}$ form EXP_{cw}'s. For example, see Fig. 4(a). An *EXP_{ccw} couple* (2EXP_{ccw} for short) is shown in Fig. 4(b). Pattern τ and reverse τ are obtained by combining an EXP_{cw} and EXP_{ccw} with an EXPS_{sym} that share a reflex vertex, as shown in Fig. 4(c) and (d), respectively, where w is trapped by trap $\{u, v\}$. Their V-diagrams are shown in Fig. 5.

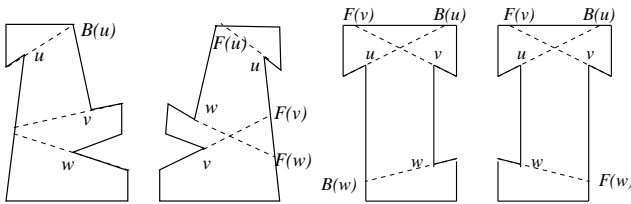


Figure 4: (a) 2EXP_{cw}; (b) 2EXP_{ccw}; (c) Pattern τ ; (d) Reserve τ .

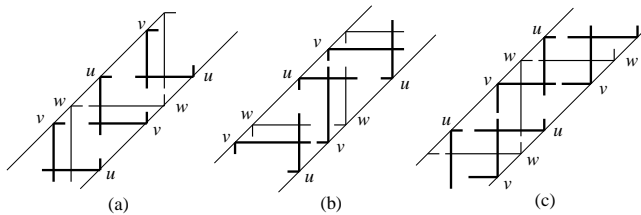


Figure 5: Skeleton V-diagrams for: (a) 2EXP_{cw}; (b) 2EXP_{ccw}; (c) Pattern τ .

We can see from Fig. 5(c) that any polygon that contains the τ pattern is not searchable [8], since sweeping

does not terminate.² In other words, pattern τ is *forbidden*. Similarly for the reverse τ pattern. Each point or edge in $\partial P(v, w)$ in Fig. 4(a) (resp. (b)) is said to be *covered* by the 2EXP_{cw} (resp. 2EXP_{ccw}).

We now introduce three conditions for a given polygon P to be *not* searchable by a boundary 1-searcher.

- FP1: P contains pattern τ or reverse τ .
- FP2: Each edge of P is trapped or covered by a 2EXP_{cw}.
- FP3: Each edge of P is trapped or covered by a 2EXP_{ccw}.

Theorem 3 *A given polygon P is searchable by a boundary 1-searcher if and only if it satisfies none of the three conditions FP1, FP2 or FP3.*

Proof. *Necessity:* Let us apply the sweeping operation to the V-diagram of polygon P satisfying any of these conditions. If P contains the τ or reverse τ pattern, the black fluid keeps flowing, i.e., the area that gets blackened will be infinite. See Fig. 5(c). Similarly, if FP2 or FP3 holds, then black fluid keeps flowing. The necessity thus follows from Theorem 2.

Sufficiency: Assume that the given polygon P satisfies none of the conditions, FP1, FP2, or FP3. We want to show that the parts of the gray region that are painted black are finite regardless of where we draw the initial sweep line $y = c_0$.

Let e be an edge of P that is neither trapped nor covered by a 2EXP_{ccw}, and let d be an edge of P that is neither trapped nor covered by a 2EXP_{cw}. Such e and d always exist by our assumption. In the V-diagram, both e and d appear infinitely many times. We pay attention to one of their instances such that $d + |\partial P| < e$ in the V-diagram. We want to show that painting due to the sweeping operation eventually stops, i.e., the blackened sections of line $y = c$ vanish as c is decreased. Start a sweep above e in the V-diagram, and consider any SE barrier S' that $y = c$ encounters after sweeping past e . We claim that the intersection of $y = c$ and S' does not get black *when S' is first encountered* by the line $y = c$ as it moves down, and moreover, there is no SE barrier S'' lying below S' that gets black when it is first encountered. The reason is as follows:

- (i) If black fluid is passed to S' via an SE barrier that touches the start line \mathcal{S} above S' , it implies e is trapped, a contradiction. (See Fig. 6(a).)
- (ii) Assume that black fluid is passed to S' via an NW barrier N which intersects S' . Such an N cannot intersect a blackened SE barrier above e , since e is not covered by any 2EXP_{ccw}. (See Fig. 6(b).)

²The V-diagram for Fig. 4(d) is “symmetric” to Fig. 5(c).

(iii) So, suppose such an N intersects a blackened NW barrier. If S' intersects another SE barrier S'' , as shown in Fig. 6 (c) and (d), it implies the existence of the τ or reverse τ pattern, a contradiction.

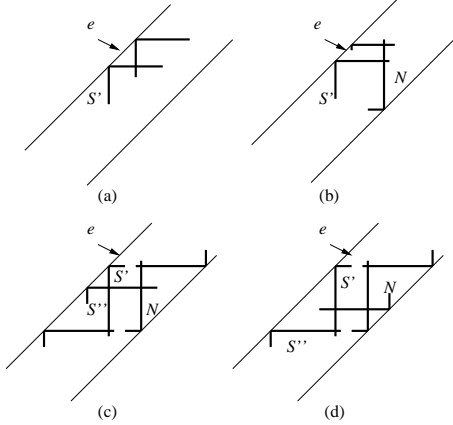


Figure 6: Illustration for the proof of Theorem 3.

It follows that there is no SE barrier lying below edge e that becomes black at the point where it touches the start line \mathcal{S} . (A part of such an SE barrier may still be blackened, as we see below.) Therefore, black fluid is propagated from an NW barrier to another NW barrier.

Assume now that NW barrier N that touches line \mathcal{G} at a point below edge d is still blackened. There are two possibilities.

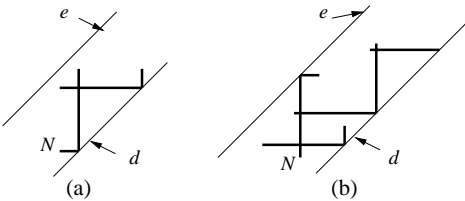


Figure 7: More illustration for the proof of Theorem 3.

- (a) Black fluid was passed to N from an NW barrier lying above d , as in Fig. 7(a). This is impossible, since this implies that d is trapped.
- (b) Black fluid was passed to N from an SE barrier, which in turn was blackened via an NW barrier lying above d . This is also impossible, since this implies that d is covered by a 2EXPCw as in Fig. 7(b).

It follows that black fluid flow stops after edge d . \square

5 Conclusion and Discussion

We identified three forbidden patterns and proved that the polygon is searchable by a boundary 1-searcher if

and only if none of them is present. The proof of correctness has been shortened drastically, compared with past work on similar topics, e.g., [5, 6, 9], thanks to high-level concepts of the V-diagram and the sweeping operation. Since our objective is characterization, we could use a higher-level, less problem-specific and more general approach. The time complexity of the test implied by Theorem 3 is $O(n \log n)$.

Three forbidden patterns, called $C1$, $C2$ and $C3$ are used to characterize the set of polygons searchable by a boundary 1-searcher in [9]. Condition $C1$ refers to the pattern first identified in [8]: There are three vertices such that the shortest path between two them is not visible from the third. Note that patterns τ and reverse τ are special cases of this forbidden pattern.

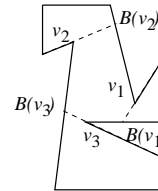


Figure 8: Polygon satisfying our FP2 and $C1$ in [9].

FP2 and FP3 are similar to $C2$ and $C3$ in [9], respectively, but they are not identical. Fig. 8 shows an example illustrating the difference. Condition $C2$ requires $v_3 \prec B(v_1) \prec v_2$ but FP2 only requires that v_1 and v_2 form an EXPCw. Therefore, the polygon in Fig. 8 satisfies FP2, but it doesn't satisfy $C2$. It is easy to check that this polygon does satisfy condition $C1$.

We could have proved that the absence of $C1$ – $C3$ is sufficient for a polygon to be searchable by a boundary 1-searcher. We adopted FP1–FP3, since the sufficiency proof requires slightly less work with them than with $C1$ – $C3$.

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