Point Lattices in Sampling Theory

Multidimensional Sampling Theory

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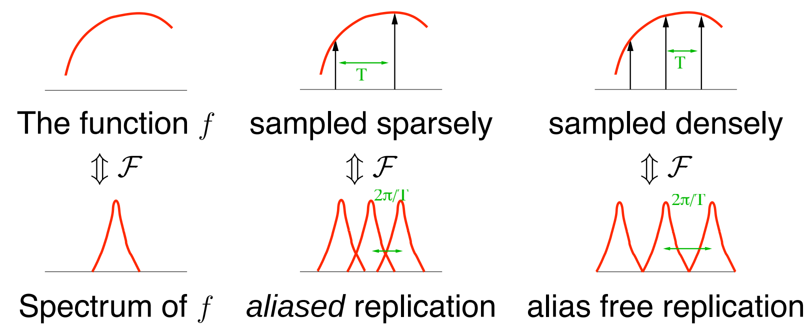
Sampling Theory in 1D

- Continuous-domain function
- Fourier domain representation



Sampling in 1D

- Space Domain: Periodic sampling with period T

$$f_s[k] = f(Tk)$$

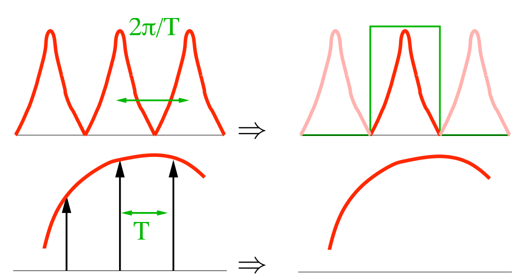
- Fourier Domain: Periodic replication of spectra with period $\frac{2\pi}{T}$

$$\hat{f}_s(\omega) = \sum_{k \in \mathbb{Z}} \hat{f}(\omega - \frac{2\pi k}{T})$$

- Discrete samples correspond to periodic replication with a reciprocal period

Reconstruction in 1D

- Best approximation in the space of bandlimited functions
- Space domain: Get a continuous function back from discrete samples
- Fourier domain: Remove the periodicity of the spectrum. Diminish the replica

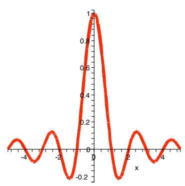


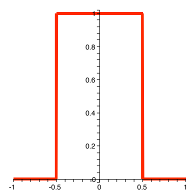
Ideal reconstruction in 1D

- Best convolution kernel for reconstructing *into the space of bandlimited functions*

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x} \qquad (1)$$

- *Infinite support in space domain!*
- *Fourier domain: Box function*

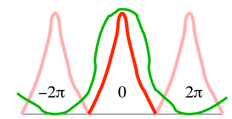




The sinc function   *the Box function*

Compact support kernels

- Compact support means infinite support in Fourier domain
- Diminish the replicas by zeroing out the replicas located at aliasing frequencies: $\ldots, \frac{-4\pi}{T}, \frac{-2\pi}{T}, \frac{2\pi}{T}, \frac{4\pi}{T}, \ldots$.
- Strang-Fix Vanishing moments: The order of zeros at the aliasing frequencies determines the smoothness of reconstruction

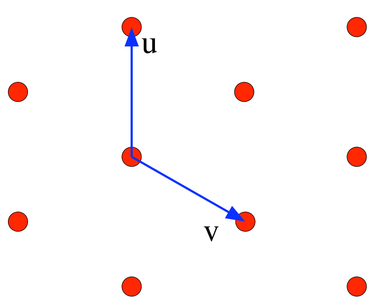


The zero crossings are important at *aliasing* frequencies

Multidimensional Sampling Theory

- Space Domain: Periodic sampling with period *matrix* $\boldsymbol{M} = [\boldsymbol{uv}]$

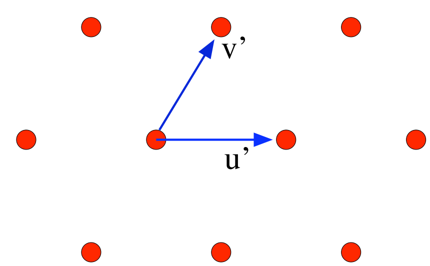
$$f_s[\boldsymbol{k}] = f(\boldsymbol{Mk}), \boldsymbol{k} \in \mathbb{Z}^n$$



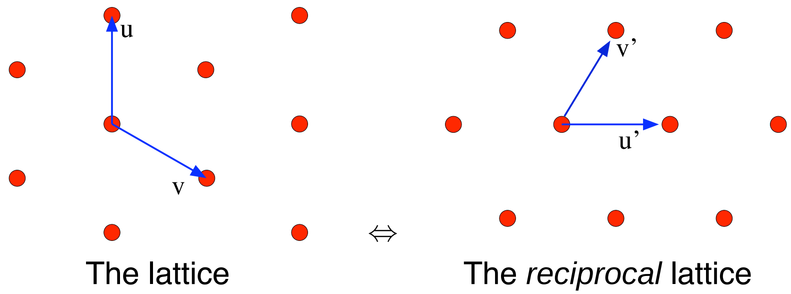
Multidimensional Sampling Theory

- Fourier Domain: Periodic replication of spectra with the period matrix $2\pi\boldsymbol{M}^{-\top} = [u'v']$.

$$\hat{f}_s(\boldsymbol{\omega}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \hat{f}(\boldsymbol{\omega} - 2\pi\boldsymbol{M}^{-\top}\boldsymbol{k})$$



Multidimensional Sampling Theory



Replication of Spectrum

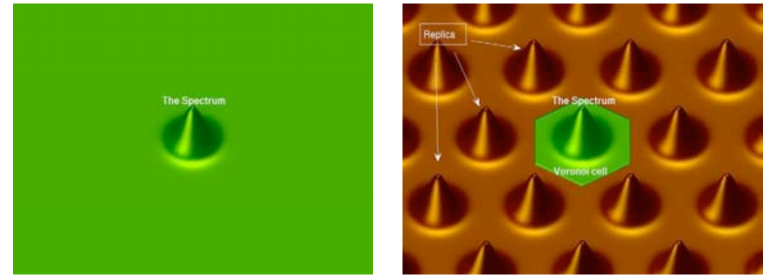


Figure 1: The Effect of sampling with $M = [uv]$ in the Fourier domain is replication on $2\pi M^{-\top}$, i.e. the *reciprocal lattice*

Reconstruction/Interpolation

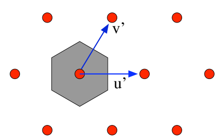
- Reconstruction in the space of bandlimited functions
- Sampling results in a periodic replication in the Fourier domain
- Reconstruction = inverse sampling
- Fourier domain: Removing the replica

Convolution based reconstruction

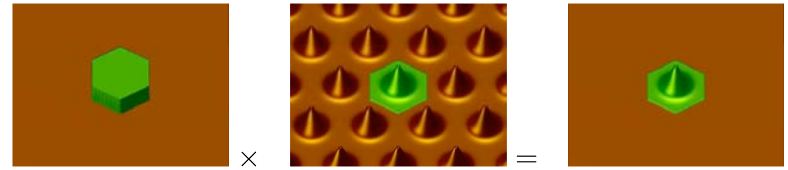
- Convolution based reconstruction = multiplication by a function that eliminates all but the main copy of the spectrum
- The replica are located on the *reciprocal, dual or polar* lattice.
- The Voronoi cell of the reciprocal lattice is where the primary copy spectrum is located (also known as Brillouin Zones).

The Ideal Bandlimited Reconstruction

- sinc function for the lattice M is defined to be the function whose Fourier transform is the characteristic function of the Voronoi cell of the lattice dual to M.
- The Fourier transform of the sinc function is constant on the interior of the Voronoi cell of the dual lattice and is zero everywhere else.
- Fourier transform of sinc_M: Characteristic function of the Voronoi cell of the reciprocal lattice



Ideal Bandlimited Reconstruction



- Fourier domain: $\mathcal{F}\{\mathrm{sinc}_M\}(\boldsymbol{\omega}) \times \hat{f}_s(\boldsymbol{\omega}) = \hat{f}_r(\boldsymbol{\omega})$
- Space domain: $\mathrm{sinc}_M(\boldsymbol{x}) * f_s(\boldsymbol{x}) = f_r(\boldsymbol{x})$

Other reconstruction kernels

- Reconstruction kernels with compact support need to have unbounded support in the Fourier domain
- One measure of accuracy of such kernels is by how much they diminish the influence of the replicas
- Strang-Fix result in the multidimensional case: the smoothness of reconstruction is determined by the number of vanishing moments.
- Vanishing moments = the order of zero roots at the centers of the replicas = at the dual lattice points

Compact support reconstruction kernels

- Tensor product of one dimensional compact support reconstruction kernels
- Radial based interpolation
- Multidimensional B-splines: successive convolutions of Characteristic function of the Voronoi cell of the sampling lattice, [See Hexsplines].
- Box splines: Projection of higher dimensional boxes
- Simplex and Polyhedral splines: Projection of higher dimensional simplices and polyhedra.

Point Lattices in Sampling Theory

From Sphere Packing to the Optimal Sampling Lattice

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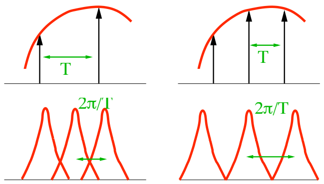
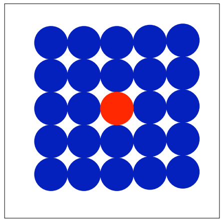
The uncertainity principle

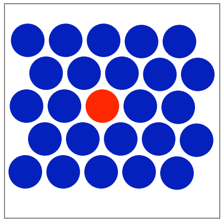
- The closer the samples are together, the farther replicas are from each other
- The farther the samples are from each other, the closer the replicas are together
- Optimal uniform sampling in 1D: bring the replicas of the spectrum as close to each other as possible



Optimal lattice sampling

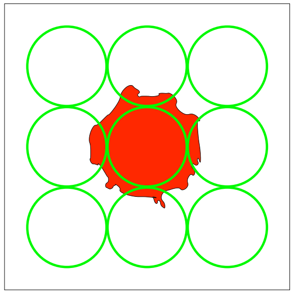
- Isotropic spectrum ⇒ uniform resolution on all orientations
- Spherical support of the Fourier representation
- Optimal lattice sampling in multidimensions: pack the replicas of the spectrum as tight as possible to each other
- Pack the spherical shape of the spectrum as dense as possible ⇒ Sphere packing!

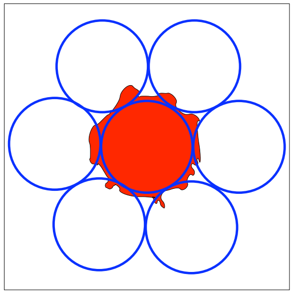




Optimal lattice sampling

- Same sampling density ⇒ Capture more information!
- In the Fourier domain cover larger part of the spectrum
- Hexagonal sampling captures higher frequencies with equal sampling density





Kepler's problem

- What is the tightest arrangement of equal sized spheres in 3D?
- Motivated by dense packing of cannon balls on a battleship
- Grocer's method: Face Centered Cubic packing

Images Courtesy of Max Wardetzky http://www.zib.de/wardetzky/kepler/index.html

- Packing density: $\frac{\text{volume of inscribing sphere}}{\text{volume of the Voronoi cell}}$

Kepler Conjecture

- Can the packing density of FCC be bettered by another arrangement?
- Gauß proved that we can not if we want the arrangement still to be a lattice
- Harder to prove without the assumption of lattice packing
- Abstracted in the Hilbert's 18^{th} problem
- Hales' found a computer aided proof in 1998.

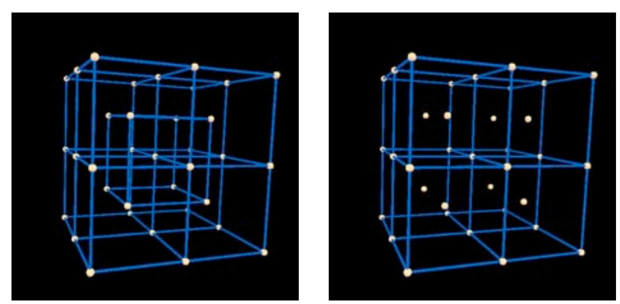
The Face Centered Cubic packing:

Sphere packing in 3D

- The FCC packing is as dense as it gets!
- The study of point lattices and sphere packing are integral parts of each other
- The theory has fruited various other fields:
  - Quantization
  - Numerical integration
  - Design of error correcting codes, coding theory, Leech lattice
  - Sampling Theory [Covered in this course]

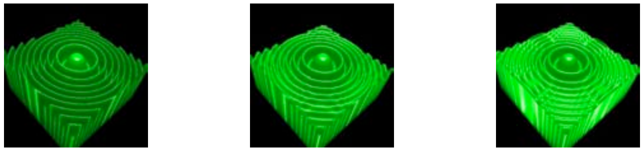
Sphere packing in Sampling Theory

- Densest packing in Fourier domain
- Sparsest pattern for distribution of samples in space domain
- The Dual lattice!
- The Body Centered Cubic (BCC) lattice:



Optimality of the BCC

- Given the same number of samples with a Cartesian sampling, the BCC sampling captures 30% more information:



Original   Body Centered Cubic   Cartesian

- Information contained in the Cartesian sampled data can be represented with 70% of the samples, if sampled on the BCC lattice