CMPT409/815: Advanced Algorithms

Homework Assignment 4

Instructor: Igor Shinkar *Due date: November 21, 2019*

Instructions: Submit either in class (hard copy) or to Coursys (if scanned, make sure it is good quality).

Graph coloring:

Question 1 (20 points) *Suppose we have a polynomial time algorithm that for any input n-vertex graph G that is 3-colorable finds an independent set of size $\geq \gamma n$ for some $\gamma > 0$. Design a polynomial time algorithm that given as input a n-vertex graph G that is 3-colorable finds a legal $O(\frac{\log(n)}{\gamma})$-coloring of G.*

Question 2 (20 points) *Design a polynomial time algorithm that gets a 4-colorable graph G and outputs a legal $\tilde{O}(n^{4/7})$ coloring of G.*

(Hint1: show first an algorithm that colors a 3-colorable graph with $\tilde{O}(n^{1/4})$ colors.)

(Hint2: to prove hint1 use the algorithm we saw in class that colors a 3-colorable graph with $\tilde{O}(\Delta^{1/3})$ colors + the algorithm that find an independents set of size $\Omega(n/\Delta)$)

Question 3 (20 points) *Given an n-vertex graph $G = (V, E)$ consider the following SDP*

$$\begin{aligned} \textit{find a feasible solution:} \quad & v_1, \dots, v_n \in \mathbb{R}^n \\ & \langle v_i, v_j \rangle = -\frac{1}{k-1} \quad \forall (i,j) \in E \\ & \|v_i\| = 1 \end{aligned}$$

Prove that if G is k-colorable, then the SDP has a feasible solution.

(Hint: prove it first for $k = 4$)

Fourier analysis of the boolean functions

Question 4 (20 points) *Prove that if a boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is 0.1-close to some linear function L, then it is at least 0.4-far from all other linear functions.*

(Hint: Prove that for any two distinct linear functions L_1, L_2 it holds that $\Pr_{x \in \{0,1\}^n}[L_1(x) = L_2(x)] = 1/2$.)

Question 5 (20 points) *Let $f\colon \{0,1\}^n \to \{0,1\}$ be a boolean function, and let $C_{1/2+\delta}(f)$ be the set of all linear functions L such that $\Pr[f(x) = L(x)] > 1/2 + \delta$. Prove that $|C_{1/2+\delta}(f)| \leq O(1/\delta^2)$ for all f and all $\delta \in (0, 0.1)$.*

(Hint: Look at the Fourier coefficients of f.)