CMPT881: Special Topics in Theoretical Computer Science

Homework assignment 3

Instructor: Igor Shinkar *Due date: March 28, 2019*

Question 1 *Design an algorithm that gets a 4-colorable graph G and outputs in polynomial time a legal $\tilde{O}(n^{4/7})$ coloring of G. You may use as a subroutine the algorithm we saw in class that colors a 3-colorable graph with $\tilde{O}(n^{1/4})$ colors.*

Question 2 *Given an n-vertex graph $G = (V, E)$ consider the following SDP*

$$\text{find a feasible solution:} \quad v_1, \ldots, v_n \in \mathbb{R}^n$$
$$\langle v_i, v_j \rangle = -\frac{1}{k-1} \quad \forall (i,j) \in E$$
$$\|v_i\| = 1$$

Prove that if G is k-colorable, then the SDP has a feasible solution.

[Hint: prove it first for $k = 4$]

Question 3 *Prove that if a boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is 0.1-close to some linear function L, then it is at least 0.4-far from all other linear functions.*

[Hint: Prove that for any two distinct linear functions L_1, L_2 it holds that $\Pr_{x \in \{0,1\}^n}[L_1(x) = L_2(x)] = 1/2$.]

Question 4 *Let $f\colon \{0,1\}^n \to \{0,1\}$ be a boolean function, and let $C_{1/2+delta}(f)$ be the set of all linear functions L such that $\Pr[f(x) = L(x)] > 1/2 + \delta$. Prove that $|C_{1/2+delta}(f)| \leq O(1/\delta^2)$ for all f and all $\delta \in (0, 0.1)$.*

[Hint: Look at the Fourier coefficients of f.]

Question 5 *We saw in class that for any $\varepsilon \in (0, 0.49)$ if a boolean function $f\colon \{0,1\}^n \to \{0,1\}$ satisfies $\Pr_{x,y \in \{0,1\}^n}[f(x) + f(y) = f(x+y)] > 1 - \varepsilon$, then f is ε-close to some linear function.*

Prove the converse of this statement (up to a constant factor). Specifically, prove that for $\varepsilon \in (0, 0.1)$ if f is close to some linear function, then $\Pr[f(x) + f(y) = f(x+y)] > 1 - 3\varepsilon$.