A Basic Representation and Reasoning System

CMPT 411/721
Reasoning with Definite Clauses

• We next define a simple KR system based on **definite clauses**.
• A definite clause can be thought of as a simple rule, with no negation in the head or body of the rule.
• This language is quite restricted, but we can still define entailment and inference, etc.
• In general, a KB will consist of **facts** and **rules**, and we will be interested in deriving other facts.
The Definite Clause Language: Vocabulary

- Assume that an agent’s knowledge is made up of two components:
  - A database of facts about the domain (or ground atomic formulas)
    E.g. Mother(jane, paul), Male(arvind).
  - A collection of rules (or definite clauses)
    E.g.
    Parent(X, Y) \iff Mother(X, Y)
    Gf(X, Y) \iff Father(X, Z) \land Parent(Z, Y)

- Note that implication is written in the reverse direction from normal.
- Variables are implicitly universally quantified.
- Variables are local to a clause.
The Definite Clause Language: Vocabulary

The vocabulary of our language is made up of:

1. Logical symbols: “(”, “)” , “,” , “⇐”, “∧”, “.”
   • Note that ¬ and ∨ aren’t included.

2. Non-logical symbols:
   • Constants, predicate symbols, function symbols
     • Uncapitalised strings.
     • Meaning of a string is implicit in its use.
     • E.g.: johnQsmith, bestFriendOf.
   • Variables
     • Written as capitalised strings.
     • E.g.: X, X₁, Variable.
The Definite Clause Language: Syntax

As in FOL, the language expresses

- *terms* that denote objects in the domain and
- *formulas* that make assertions about the domain.
The Definite Clause Language: Terms

A term is either

- a variable,
- a constant, or
- an expression of the form $f(t_1, \ldots, t_n)$ where $f$ is a function symbol, and each $t_i$ is a term.
The Definite Clause Language: Formulas

- Formulas are defined as follows:
  - An atomic formula (atom) is of the form $p$ or $p(t_1, \ldots, t_n)$ where $p$ is a predicate symbol, and each $t_i$ is a term.
  - A body is of the form $a_1 \land \cdots \land a_n$ where each $a_i$ is an atom.
  - A definite clause is of the form $a_\leadsto b$ where $a$, the head, is an atom and $b$ is a body.

- A knowledge base is a set of definite clauses.
- Although it isn’t part of the language, a query is conventionally written in the form $?b$. where $b$ is a body.
Example

Example

• (Ground) atomic formulas:
  \[
  \begin{align*}
  &\text{father}(ian, sue) \\
  &\text{father}(fred, chris) \\
  &\text{mother}(michelle, chris) \\
  &\text{num}(0)
  \end{align*}
  \]

• Definite clauses:
  \[
  \begin{align*}
  &\langle \text{the above atomic formulas} \rangle \\
  &gf(ian, chris) \leftarrow \text{father}(ian, fred) \land \text{father}(fred, chris) \\
  &gf(X, Y) \leftarrow \text{father}(X, Z) \land \text{father}(Z, Y) \\
  &\text{num}(s(N)) \leftarrow \text{num}(N) \\
  &\text{num}(X) \leftarrow \text{father}(X, Y)
  \end{align*}
  \]
Semantics

- Meaning is attached to symbols as in FOL.
- An interpretation is a pair $\mathcal{I} = \langle D, I \rangle$ where
  1. $D \neq \emptyset$ is the domain.
  2. $I$ is a mapping that assigns
     - to each constant: an element of $D$
     - to each $n$-ary function symbol: a mapping from $D^n \Rightarrow D$ and
     - to each $n$-ary predicate symbol: a subset of $D^n$
       (0-ary predicate symbols are assigned true or false in an interpretation.)
We first give a semantics for variable-free or \textit{ground} expressions:

- Each ground term denotes an individual in the domain:
  - Constant $c$ denotes the individual $I(c)$ in $\mathcal{I}$.
  - $f(t_1, \ldots, t_n)$ denotes the individual $I(f)(t'_1, \ldots, t'_n)$ in $\mathcal{I}$, where $t'_i$ is the individual denoted by $t_i$ (i.e. $t'_i = I(t_1)$).
Semantics (continued)

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  - Each ground atomic formula is either *true* or *false* in an interpretation.
    - Atom $p(t_1, \ldots, t_n)$ is *true* in $\mathcal{I}$ if $\langle t'_1, \ldots, t'_n \rangle \in I(p)$ where $t'_i = I(t_1)$; otherwise *false*. 
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  • **Truth** in interpretation \(I\) is defined by:
    
    • \(P \land Q\) is true iff \(P\) is true and \(Q\) is true.
    • \(Q \iff P\) is true iff \(P\) is false or \(Q\) is true.
  
  ✨ At this point every variable-free formula is true or false in an interpretation.
Semantics: Variables

A *variable assignment* $\nu$ is used to define the semantics of formulas with variables.

- As with FOL, a variable assignment is a function from the set of variables into the domain.
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- A clause $C$ with variables is false in interpretation $\mathcal{I}$ just if there is a variable assignment $\nu$ under which the clause is false.

Recall: Variables are local to a clause.

Recall: Variables in a clause are regarded as universally quantified.

I.e. $C$ is true for every variable assignment.
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- A clause \( C \) with variables is false in interpretation \( I \) just if there is a variable assignment \( \nu \) under which the clause is false.
  - Recall: Variables are local to a clause.
  - Recall: Variables in a clause are regarded as universally quantified.
- A clause \( C \) with variables is true in \( I \) just if it isn’t false.
  - I.e. \( C \) is true for every variable assignment.
Semantics: Entailment

Finally:

- A set of clauses $C$ is *true in an interpretation* $\mathcal{I}$ iff every element of $C$ is true in $\mathcal{I}$.
  - $\mathcal{I}$ is a *model* of $C$. 

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  - $\mathcal{I}$ is a *model* of $C$.

- If $S$ is a set of clauses and $g$ is an *atom* or *conjunction of atoms*, then $g$ is *logically entailed* by $S$, written $S \models g$, iff $g$ is true in every model of $S$.
  - I.e. every model of $S$ is a model of $g$.
  - So the same definition as in FOL, but in a restricted language.
    - Note the restricted form of $\models$.

- The relation $\models$ says nothing about computation, proof, derivation, etc.
  - $\models$ just says what is true, given that other things are true.
User’s View of Semantics

Recall that the idea behind our use of logic is that we have a particular domain in mind to represent, the *intended interpretation*.

- We choose denotations for our symbols with respect to this domain and write, as clauses, what is true in that world.
  - I.e. we *axiomatise* our domain.
- When the system gives us a logical consequence of our axioms we can interpret this answer with respect to our intended interpretation.
- Again, this is no different than in FOL, except that we have a limited language.
Semantics and Logical Consequence

- The computer does not have access to the intended interpretation, but only to the axiomatisation.
- Given an appropriate *inference procedure*, the computer will be able to tell whether some statement is a logical consequence of the axioms.
  - If it is a logical consequence, then it is true in the intended interpretation (assuming the axioms are correct).
Queries and Answers

- As with FOL, we build a formal description of the world in order to ask questions about it.
  - Want to ask about information *implicit* in the knowledge base.
  - If we were just interested in *retrieval* of explicit information (as in a database) we wouldn’t need a formal model.

- A *query* defines the syntax by which we ask whether something is a logical consequence of the knowledge base.

- Queries can be represented syntactically as *?body*. 
Queries and Answers

• A query is a question to which we want the answer:
  • yes if the query is a consequence of the knowledge base and
  • no if the query is not a consequence of the knowledge base.

• No doesn’t mean that the query is false in the intended interpretation.

• Rather no means that we don’t know whether it is true in the intended interpretation.
Queries and Answers

- One way of treating queries, is that for \( ?\text{body} \), it is as if we added a clause
  \[ \text{answer} \iff \text{body}. \]
to the knowledge base (for new atom \textit{answer}).

- We then try to show that \textit{answer} is a logical consequence of the KB.

- If we can show that \textit{answer} is a logical consequence, then so is \textit{body}.

- This scheme provides a uniformity wrt query answering; as well it allows us to express \textit{answers} via an \textit{answer predicate} (later).
Variables

• Recall: When a clause contains variables, that clause is true in an interpretation only if it is true for every possible value of the variables.

• So if $X$ appears in clause $C$ then
  
  $C$ is true in an interpretation

  means that

  $C$ is true no matter what individual is denoted by $X$.

• For example, for

  \[ gf(X, Y) \leftarrow father(X, Z) \land parent(Z, Y). \]

  to be true, it must be true no matter what individuals are denoted by $X$, $Y$ and $Z$. 
Variables

One potentially confusing point is the following:

Variables that appear only in the body of a clause can be considered to be universally quantified at the level of the clause, and existentially quantified in the body.

For example, if we use explicit quantifiers $\forall X$ and $\exists X$, then we have that

$$\forall X \forall Y \forall Z (gf(X, Y) \iff father(X, Z) \land parent(Z, Y))$$

means the same thing as

$$\forall X \forall Y (gf(X, Y) \iff \exists Z (father(X, Z) \land parent(Z, Y)))$$
Variables and Queries

Variables in queries are handled by our previous translation.

• Example: \( \text{?gf}(X, \text{ian}) \) can be translated to:

\[
\text{answer} \leftarrow \text{gf}(X, \text{ian})
\]

Or, using the second reading from the previous slide:

\[
\text{answer} \leftarrow \exists X \text{ gf}(X, \text{ian})
\]

• I.e \( \text{answer} \) is true if there is some \( X \) who is the grandfather of \( \text{ian} \).
Variables and Queries

- Typically we want to know not just *whether* there is a grandfather of Ian, but *who* the grandfather of Ian is.
- For this we translate the query \(?gf(X, ian)\) to the *answer clause*

\[
answer(X) \iff gf(X, ian)
\]

- In general, if the query is \(B\) with free variables \(X_1, \ldots, X_n\), then the answer clause is

\[
answer(X_1, \ldots, X_n) \iff B
\]

- The aim now is to determine which *instance* of \(answer(X_1, \ldots, X_n)\) is a consequence of the KB.
Inference

• So far we have specified what we would like an answer to be, but not how it can be computed.
  • I.e. we have just considered conditions under which a clause is true in an interpretation.

• Now we want to explore means by which logical consequences of a set of clauses can be computed solely on the basis of their form, and without considering interpretations.
  • I.e. we want to determine an inference procedure or proof procedure for our clause language.

• For a proof procedure, we write
  \[ S \vdash g \]
  to mean \( g \) can be derived from \( S \).
A proof procedure can be judged by whether it computes what it is meant to compute.

As before:

- A proof procedure is **sound** with respect to a semantics if everything derivable is justified by the semantics. That is
  \[ \text{If } S \vdash g \text{ then } S \models g. \]

- A proof procedure is **complete** with respect to a semantics if there is a proof for every logical consequence of the clauses. That is
  \[ \text{If } S \models g \text{ then } S \vdash g. \]
A Bottom-up Proof Procedure

- Idea: Starting from the initial facts and rules in the KB, derive further facts.
  - Also called *forward chaining*.
- The procedure is based on a *rule of derivation*, a generalised rule of “modus ponens”:
  
  If \( h \iff b_1 \land \cdots \land b_m \) is a clause, and each \( b_i \) has been derived, then \( h \) can be derived.

- As a base case, we have that every fact is (trivially) derived.
- We consider the variable-free case first.
A Bottom-up Proof Procedure

Procedure:
$C := \{\}$;
repeat
    choose $r \in S$ such that
    $r$ is ‘$h \iff b_1 \land \cdots \land b_m$’
    $b_i \in C$ for all $i$, and
    $h \notin C$;

    $C := C \cup \{h\}$

until no more choices

We write $S \vdash g$ if $g \in C$ at the end of the procedure.
Example

\[ a \leftarrow b \land c \]
\[ b \leftarrow d \land e \]
\[ c \leftarrow e \]
\[ d \]
\[ e \]
\[ f \leftarrow a \land g \]

Obtain:

\{ d, e, c, b, a \}
Example

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\begin{align*}
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Obtain: \( \{d, e, c, b, a\} \).
Properties of the Procedure:

1. Soundness: Every atom in $C$ is a logical consequence of $S$.

Exercise: Prove the above items.
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   - The algorithm is linear in the size of the KB (so long as we can index the clauses so that the inside loop can be carried out in constant time).

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4. **Fixed Point**: The final \( C \) is called a *fixed point*.
   - Let \( \mathcal{I} \) be the interpretation in which every atom in the fixed point is *true* and every atom not in the fixed point is *false*. Then: \( \mathcal{I} \) is a model of \( S \).
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A Top-down Proof Procedure

An alternative proof method is to search backwards from the query to determine whether it is a logical consequence of $S$.

Also called \textit{backward-chaining} inference
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- We define *definite clause resolution* for the ground case, then consider the general case with variables.
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  - An *answer clause* is of the form

    \[
    \text{answer} \iff a_1 \land \cdots \land a_m
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- An *answer clause* is of the form

  $$\text{answer} \leftarrow a_1 \land \cdots \land a_m$$

- A *resolution* of the above clause with the clause

  $$a_1 \leftarrow b_1 \land \cdots \land b_n$$

  is the answer clause

  $$\text{answer} \leftarrow b_1 \land \cdots \land b_n \land a_2 \land \cdots \land a_m$$
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  $$\text{answer} \leftarrow b_1 \land \cdots \land b_n \land a_2 \land \cdots \land a_m$$

- An *answer* is an answer clause with no body
A Top-down Proof Procedure

- A *derivation* of a query \(?q_1 \land \cdots \land q_k\) from rules \(S\) is a sequence of answer clauses \(\gamma_0, \ldots, \gamma_p\) such that
  1. \(\gamma_0\) is the answer clause:

\[
answer \leftarrow q_1 \land \cdots \land q_k,
\]

  2. \(\gamma_i\) is obtained by resolving \(\gamma_{i-1}\) with a clause in \(S\), and
  3. \(\gamma_p\) is an answer.

This is just proposition resolution under a (slightly) different guise and in a simpler language.

Note that it implements a set of support strategy.
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- This is just proposition resolution under a (slightly) different guise and in a simpler language.
- Note that it implements a *set of support* strategy.
A Top-down Interpreter:

\[
solve(q_1 \land \cdots \land q_k):
\]

\[
an \leftarrow \{(answer \leftarrow q_1 \land \cdots \land q_k)\}
\]

\[repeat\]

\[
choose C \text{ from } S
\]

\[ac := resolve(ac, C)\]

\[until \ ac \ is \ an \ answer\]

- Note that in this case, the nondeterministic “choose” relies on guessing the “right” clause for resolution.
- The differing types of nondeterminism (as in the bottom-up and top-down procedures) have been called \textit{select} vs. \textit{choose} nondeterminism.
Aside: Select and Choose Nondeterminism

Select nondeterminism:

- For *select* nondeterminism, if the language is finite and there are no variables, then it doesn’t matter what nondeterministic choice you make.
- E.g. for the bottom-up procedure, eventually every derivable atom will be derived.
- For variables you have to be more careful.

Choose nondeterminism:

- For *choose* nondeterminism, one has to make the “right” nondeterministic choice.
- Just because one choice doesn’t lead to an answer doesn’t mean other choices will be futile.
- So here we also have a search problem.
Example

Example

\[ a \leftarrow b \land c \]
\[ b \leftarrow d \land e \]
\[ c \leftarrow e \]
\[ d \]
\[ e \]
\[ f \leftarrow a \land g \]
\[ ?a \]

One sequence of assignments to \textit{answer} is:

\[ answer \leftarrow a \]
Example

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Example

One sequence of assignments to answer is:

```
answer \leftarrow a
answer \leftarrow b \land c
answer \leftarrow d \land e \land c
answer \leftarrow e \land c
answer \leftarrow c
answer \leftarrow e
```
1 When we have derived the answer, we can read a bottom-up “proof” in the opposite direction.
   • Also every top-down derivation corresponds to a bottom-up proof and every bottom-up proof has a corresponding top-down derivation.

2 The preceding equivalence can be used to show the soundness and completeness of the derivation procedure.
Variables and Substitutions

Variables and substitutions are handled exactly as in FOL:

- An *instance* of a clause is obtained by uniformly substituting terms for variables in the clause.
- If a clause is true in an interpretation then any instance will also be true in that interpretation.
- A *substitution* is a set of statements of the form $v/t$, where $v$ is a variable and $t$ is a term.

Problem: There may be infinitely many instances of a clause if we have function symbols.

- E.g.: $num(0)$, $num(s(0))$, $num(s(s(0)))$, ...
Variables and Substitutions

• A substitution is in normal form if each variable on the left-hand side appears nowhere else in the substitution.
  • Assume all substitutions are in normal form.

• A substitution $\theta$ applied to an expression $e$ is an expression $e\theta$ which is like $e$, but with all instances of variables on the lhs of a "/" replaced by the term on the rhs.

• E.g., applying

  \[
  \theta = \{ X/Y, Z/f(U) \}
  \]

  to

  \[
  p(X, Y) \leftarrow q(a, Z).
  \]

  is the instance

  \[
  p(Y, Y) \leftarrow q(a, f(U)).
  \]
Variables and Substitutions

Recall:

- Substitution $\theta$ is a unifier of atoms $e_1$ and $e_2$ if $e_1 \theta = e_2 \theta$.
  - E.g. $\{X/a, Y/b\}$ is a unifier of $t(a, Y, c)$ and $t(X, B, c)$.

There may be many unifiers for terms and clauses.

- E.g, $p(X, Y)$ and $p(Z, Z)$ have unifiers $\{X/b, Y/b, Z/b\}$, $\{X/f(a), Y/f(a), Z/f(a)\}$, and $\{X/Z, Y/Z\}$.
  - The third unifier is preferred because it implies the first two.
  - This is called the most general unifier, or MGU.
  - So the MGU is a unifier of two terms that is implied by all other unifiers.

MGU's exist and are unique, up to the renaming of variables.
Variables and Substitutions

Recall:

- Substitution \( \theta \) is a **unifier** of atoms \( e_1 \) and \( e_2 \) if \( e_1 \theta = e_2 \theta \).
  - E.g. \( \{X/a, Y/b\} \) is a unifier of \( t(a, Y, c) \) and \( t(X, B, c) \).

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    - \( \{X/f(a), Y/f(a), Z/f(a)\} \)
    - \( \{X/Z, Y/Z\} \).

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    \begin{align*}
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    &\{X/f(a), Y/f(a), Z/f(a)\} \\
    &\{X/Z, Y/Z\}.
    \end{align*}
    \]
  • The third unifier is preferred because it implies the first two.
    • This is called the *most general unifier*, or MGU.
    • So the MGU is a unifier of two terms that is implied by all other unifiers.

• MGU’s exist and are unique, up to the renaming of variables.
Bottom-up Procedure with Variables

- We can do the bottom-up procedure for clauses with variables, if we carry out the bottom-up procedure for all ground instances of the variables in the axioms.
- We must make certain that our procedure is *fair*, in that every usable rule is chosen eventually.
- E.g., consider:
  
  \[
  \text{num}(s(N)) \Leftarrow \text{num}(N)
  \]
  
  \[
  \text{num}(0)
  \]
  
  \[
  \text{mother}(sue, mary).
  \]

  An unfair strategy could always choose the first rule, and so never derive that \text{mother}(sue, mary).

- Our previous procedure, extended to allow variables, is sound and complete (so long as it is fair).
Bottom-up Procedure with Variables

If the domain is known to be finite, then one can handle variables by:

1. Substitute all possible instances of terms for the variables in the KB.
   - This is known as *grounding* the KB.

2. Then work with the grounded KB, using the procedure for propositional KBs.

Thus the first-order set of rules is effectively translated into a KB in propositional logic.
Top-down Procedure with Variables

Or: *Definite clause resolution with variables.*

- Suppose we have the answer clause
  
  \[ \text{answer}(t_1, \ldots, t_k) \leftarrow a_1 \land \cdots \land a_m \]

- The *resolution* of the above clause with the clause
  
  \[ a \leftarrow b_1 \land \cdots \land b_n \]

  where \( a \) and \( a_1 \) have most general unifier \( \theta \) is the answer clause:

  \[ [\text{answer}(t_1, \ldots, t_k) \leftarrow b_1 \land \cdots \land b_n \land a_2 \land \cdots \land a_m] \theta \]

- This is known as *SLD resolution*

- SLD resolution is the principal control strategy that underlies PROLOG.
Definite clause resolution with variables

- **A derivation** from rules $S$ is a sequence of answer clauses $\gamma_0, \ldots, \gamma_n$ such that
  1. $\gamma_0$ is the original answer clause.
     - If the query is $B$ with free variables $V_1, \ldots, V_k$, then $\gamma_0$ is $answer(V_1, \ldots, V_k) \leftarrow B$.
  2. $\gamma_i$ is obtained by resolving $\gamma_{i-1}$ with a clause in $S$.
  3. $\gamma_n$ is an answer.

- That is, $\gamma_n$ is of the form

  $$answer(t_1, \ldots, t_k) \leftarrow .$$

When this occurs we have an answer, $(V_1 = t_1, \ldots, V_k = t_k)$. 
Example

(from before):

\[ gf(X, Y) \iff \text{father}(X, Z) \land \text{parent}(Z, Y) \]
\[ \text{parent}(X, Y) \iff \text{mother}(X, Y) \]
\[ \text{parent}(X, Y) \iff \text{father}(X, Y) \]
\[ \text{mother}(\text{michelle}, \text{sue}) \]
\[ \text{father}(\text{ian}, \text{sue}) \]
\[ \text{mother}(\text{sue}, \text{chris}) \]
\[ \text{father}(\text{george}, \text{ian}) \]
Example

For query \(?gf(G, sue)\), we have the derivation:

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   This is resolved with the first clause in the KB with substitution \( \{ X_1/G, Y_1/sue \} \) to obtain

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2. \( answer(G) \leftarrow father(G, Z_1) \land parent(Z_1, sue) \)
   This is resolved with \( father(george, ian) \) with substitution \( \{ G/george, Z_1/ian \} \) to obtain

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For query \( ?gf(G, sue) \), we have the derivation:

1. \( \text{answer}(G) \leftarrow gf(G, sue) \)
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2. \( \text{answer}(G) \leftarrow \text{father}(G, Z_1) \land \text{parent}(Z_1, sue) \)
   This is resolved with \( \text{father}(george, ian) \) with substitution \( \{G/george, Z_1/ian\} \) to obtain

3. \( \text{answer}(george) \leftarrow \text{parent}(ian, sue) \)
   This is resolved with \( \text{parent}(X_2, Y_2) \leftarrow \text{father}(X_2, Y_2) \) with substitution \( \{X_2/ian, Y_2/sue\} \) to obtain

4. \( \text{answer}(george) \leftarrow \text{father}(ian, sue) \)

This is resolved with \( \text{father}(ian, sue) \) to obtain

\( \text{answer} = george \).
Example

For query \(?gf(G, \text{sue})\), we have the derivation:

1. \(answer(G) \leftarrow gf(G, \text{sue})\)
   This is resolved with the first clause in the KB with substitution \(\{X_1/G, Y_1/\text{sue}\}\) to obtain

2. \(answer(G) \leftarrow \text{father}(G, Z_1) \land \text{parent}(Z_1, \text{sue})\)
   This is resolved with \(\text{father}(\text{george}, \text{ian})\) with substitution \(\{G/\text{george}, Z_1/\text{ian}\}\) to obtain

3. \(answer(\text{george}) \leftarrow \text{parent}(\text{ian}, \text{sue})\)
   This is resolved with \(\text{parent}(X_2, Y_2) \leftarrow \text{father}(X_2, Y_2)\) with substitution \(\{X_2/\text{ian}, Y_2/\text{sue}\}\) to obtain

4. \(answer(\text{george}) \leftarrow \text{father}(\text{ian}, \text{sue})\)
   This is resolved with \(\text{father}(\text{ian}, \text{sue})\) to obtain

5. \(answer(\text{george}) \leftarrow \)

An answer thus is \(G = \text{george}\).
Example

Notes:

• Another answer could have been chosen by choosing different clauses for resolution.

• Some choice of clauses for resolution will lead to a dead end.

• There is an (implicit) renaming of variables for each instance/use of a clause.

• A full implementation will need to save state information in order to determine another answer.
Example: Simulating Systems

Example

Consider the domain of circuits.

- We have objects consisting of *gates* of various types, *signal values* (i.e. *on* and *off*), etc.
- We use the following predicates and functions:
  1. \( \text{gate}(G, T) \) means that gate \( G \) is of type \( T \).
     E.g.: \( \text{gate}(x_1, \text{xor}) \), \( \text{gate}(x_2, \text{xor}) \), \( \text{gate}(a_1, \text{and}) \), \( \text{gate}(a_2, \text{and}) \), \( \text{gate}(o_1, \text{or}) \).
  2. \( \text{Connected}(P_1, P_2) \) means that port \( P_1 \) is connected to port \( P_2 \).
  3. \( \text{in}(N, G) \) denotes input port \( N \) of gate \( G \).
  4. \( \text{out}(G) \) denotes the output port of gate \( G \).
  5. \( \text{out}(N, G) \) denotes output port \( N \) of circuit \( G \).
Example: Simulating Systems

- For connectivity we can assert something like:

\[
\text{value}(X, V) \iff \text{connected}(Y, X) \land \text{value}(Y, V).
\]

- To say that an \textit{and} gate has output corresponding to the conjunction of its inputs we could have:

\[
\text{value}(\text{out}(D), \text{on}) \iff \text{gate}(D, \text{and}) \land \text{value}(\text{in}(1, D), \text{on}) \land \text{value}(\text{in}(2, D), \text{on}).
\]

\[
\text{value}(\text{out}(D), \text{off}) \iff \text{gate}(D, \text{and}) \land \text{value}(\text{in}(1, D), \text{off}).
\]

\[
\text{value}(\text{out}(D), \text{off}) \iff \text{gate}(D, \text{and}) \land \text{value}(\text{in}(2, D), \text{off}).
\]
Example: Simulating Systems

Consider a full adder:

```
    1
   / \
  2   3
   \ / \
    X1

    A1
```

```
    X2
   / \
  A2  \
    O1
```

```
    Full Adder
```
Example: Simulating Systems

- We can add assertions about the values of the inputs to the circuits such as
  
  \[
  value(in(1, adder), on), \quad value(in(2, adder), off), \quad value(in(3, adder), on)\]
  
- We can determine the values of the output ports with the query

  \[
  \text{?value(out(1, adder), Out1) \land value(out(2, adder), Out2)}
  \]

- This returns \(Out1 = \text{off}\) and \(Out2 = \text{on}\).
Bottom-Up vs. Top-Down Derivations

Ask: why select top-down procedure over bottom-up, or vice versa?

- **Top-down/Backward Chaining:**
  - Query-answering
  - Directed reasoning
  - Good for user acceptability and diagnosis of KB bugs.
  - Worst-case exponential complexity
  - Harder to implement

- **Bottom-up/Forward Chaining:**
  - Gives all solutions
  - More responsive to changes in domain facts
    - E.g. Rules of form: Action ⇐ Condition
  - Linear procedure
  - More suitable for finite domains.
  - With variables, typically need to *ground* the knowledge base first