Description Logics: 

$ALC$
Topics:

1. Introduction to description logics
2. The description logic $\mathcal{ALC}$
3. Extensions to $\mathcal{ALC}$
4. A tableau algorithm for $\mathcal{ALC}$
Introduction

Description logics

• A DL is a formalism for expressing concepts, their attributes (or associated roles), and the relationships between them.
  • E.g. Person could be a concept and a role could be ParentOf.

• Can be regarded as a KR system based on a structured representation of knowledge.

• Most DLs are fragments of FOL, written in a distinct syntax.

Predecessors of DLs

• Semantic networks of the 70s
• Frame-based systems
Why Description Logics?

Ideal AI case:

- Approaches have scientific (logical) and engineering aspects
- **Scientific**: Analyse the problem formally and in detail
- **Engineering**: Get something working quickly and efficiently
- **Success**: When these two approaches coincide – efficient implementations of (formally) well-understood systems.

- Description Logic research has (arguably) reached this point
Background: Concepts, Roles, Constants

• In a description logic, there are sentences that will be true or false (as in FOL).
  • These are restricted to *subsumption* and *instance* assertions.
• In addition, there are three sorts of expressions that act like nouns and noun phrases in English:
  • *Concepts* are like category nouns: Person, Female, GraduateStudent
  • *Roles* are like relational nouns: AgeOf, ParentOf, AreaOfStudy
    • Specify attributes of concepts and their types
  • *Constants* are like proper nouns: John, Mary
• These correspond to unary predicates, binary predicates and constants (respectively) in FOL.
• Unlike in FOL, concepts need not be atomic and can have structure.
DL Knowledge Bases

An KB in a DL contains two parts:

- Define terminology: \textit{TBox}
  - E.g. \( MWD : \equiv Mother \sqcap \forall ParentOf . \neg Female \)
- Give assertions: \textit{ABox}
  - E.g. \( MWD(sue) \).
Main components of the TBox:

- **Concepts**: classes of individuals
  - E.g. *Mother*

- **Roles**: binary relations between individuals
  - E.g. $\forall ParentOf \neq Female$

- **Complex concepts using constructors**
  - E.g. $Mother \sqcap \forall ParentOf \neq Female$

- **Assertions concerning complex concepts**
  - E.g. $MWD = Mother \sqcap \forall ParentOf \neq Female$

$Mother \sqsubseteq Female$
DL Knowledge Bases: TBox

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- **Assertions** concerning complex concepts
  - E.g. $MWD \models Mother \sqcap \forall ParentOf. \neg Female$
  - $Mother \sqsubseteq Female$
DL Knowledge Bases: ABox

ABox: Assertions that individuals satisfy certain concepts and roles.

- Think of as a simple relational database.
- E.g. $MWD(Mary)$, $ParentOf(Mary, John)$. 
DL: Advantages

- Well-defined formal semantics.
- Known (and often good) complexity characteristics or implementations.
- Relatively easy to specify DL knowledge bases, in a structured hierarchical fashion.
- DLs constitute a large family of approaches.
  - Can tailor a language to a specific application.
Applications

Useful whenever a common vocabulary is important.

E.g.:

- Enhanced database systems
  - *DL-Lite*
- Medical informatics: Snomed CT, Galen
  - *EL*
- Semantic Web
  - Next generation web
  - *OWL*: W3C recommendation.

ически We’ll look at perhaps the most central DL, *ALC*. 
The Logic $\textit{ALC}$

An $\textit{ALC}$ KB contains two parts:

- Define terminology: TBox
- Give assertions: ABox
The Logic \textit{ALC}

An \textit{ALC} KB contains two parts:

\begin{itemize}
  \item Define terminology: TBox
  \item Give assertions: ABox
\end{itemize}

Main components of the TBox:

\begin{itemize}
  \item Concepts: Represent classes of individuals
  \item Roles: Represent binary relations between individuals
  \item Complex concepts using constructors
\end{itemize}

Examples:

\begin{itemize}
  \item Concept names: Person, Female
  \item Role names: ParentOf, HasHusband
  \item Individual names (in the ABox): John, Mary
\end{itemize}
The Logic $\mathcal{ALC}$: Language

Logical symbols:

- **Propositional constructors:** $\sqcap, \sqcup, \neg$
- **Other restrictions:** $\forall, \exists$
  - Note: These are different from quantifiers as seen in FOL
- $\top, \bot$
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Concept construction
- Let $C$ and $D$ be concepts and $R$ a role.
- $\neg C$, $C \sqcap D$, $C \sqcup D$ are concepts.
- $\forall R.C$, $\exists R.C$ are concepts.
Let $C$ and $D$ be concepts and $R$ a role.

- $C$ stands for a concept or set of individuals.
- $\neg C$ stands for the concept of things that are not $C$.
- $C \sqcap D$ is the concept of things that are both $C$ and $D$.
- E.g. $\text{Female} \sqcap \text{Human}$
- $C \sqcup D$ is the concept of things that are either $C$ or $D$ or both.
- E.g. $\text{Male} \sqcup \text{Female}$
- $\forall R.C$ is the concept of things such that all things that are $R$ related to it are $C$'s.
- E.g. $\forall \text{ParentOf}. \text{Female}$: things all of whose children are female
- $\exists R.C$ is the concept of things such that some thing $R$ related to it is a $C$.
- $\exists \text{ParentOf}. \text{Female}$: things with a female child
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The Logic $\mathcal{ALC}$: Knowledge Bases

Axioms (assertions) in the TBox:

- Subsumption: $C \sqsubseteq D$ where $C$ and $D$ are concepts
- Equivalence axioms: $C \equiv D$ where $C$ and $D$ are concepts

Assertions in the ABox:

- $C(a)$ where $C$ is a concept and $a$ is an individual name.
- $R(a, b)$ where $R$ is a role name, $a$ and $b$ are individual names.

DL knowledge base:

- Set of TBox statements
- Set of ABox statements
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Examples

TBox:

- \(\text{Person} \sqsubseteq \text{Animal} \sqcap \text{Biped}\)
- \(\text{Woman} \equiv \text{Person} \sqcap \text{Female}\)
- \(\text{Mother} \equiv \text{Woman} \sqcap \exists \text{ParentOf} \cdot \text{Person}\)
- \(\text{Parent} \equiv \text{Mother} \sqcup \text{Father}\)
- \(\text{Man} \equiv \text{Person} \sqcap \neg \text{Woman}\)
- \(\text{MotherWithoutDaughter} \equiv \text{Mother} \sqcap \forall \text{ParentOf} \cdot \neg \text{Female}\)
- \(\text{GrandMother} \equiv \text{Woman} \sqcap \exists \text{ParentOf} \cdot \text{Parent}\)

ABox:

- \(\text{GrandMother}(\text{Sally})\)
- \((\text{Person} \sqcap \text{Male})(\text{John})\)
Formal Semantics for Concepts and Names

Semantically, a DL can be seen as a fragment of FOL.
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An interpretation is a pair $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$

- Domain $\Delta$: non-empty set of objects
- Interpretation function $\cdot^{\mathcal{I}}$: Maps structures into the domain.
- Recall, Brachman and Levesque write this as $\mathcal{I} = \langle D, I \rangle$. 
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Then:

- $^\mathcal{I}$ maps every concept name $A$ to a subset $A^\mathcal{I} \subseteq \Delta$
- $^\mathcal{I}$ maps every role name $R$ to a binary relation $R^\mathcal{I} \subseteq \Delta \times \Delta$
- $^\mathcal{I}$ maps individual names $a$ to elements of $\Delta : a^\mathcal{I} \in \Delta$
- $\top^\mathcal{I} = \Delta$ and $\bot^\mathcal{I} = \emptyset$. 
Semantics for Complex Concepts

Assume $C$, $D$ are concepts, and $R$ is a role.

- $(\neg C)^I = \Delta \setminus C^I$
- $(C \cap D)^I = C^I \cap D^I$
- $(C \cup D)^I = C^I \cup D^I$
- $(\forall R. C)^I = \{x \mid y \in C^I \text{ for every } y \text{ s.t. } (x, y) \in R^I\}$
- $(\exists R. C)^I = \{x \mid y \in C^I \text{ for some } y \text{ s.t. } (x, y) \in R^I\}$
Semantics for Axioms and Assertions

Assume $C$, $D$ are concepts, $R$ is a role, $a$ and $b$ are individual names.
Let $\mathcal{I} = (\Delta, I^\mathcal{I})$ be an interpretation.

- $C \subseteq D$ is true in $\mathcal{I}$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$
- $C \equiv D$ is true in $\mathcal{I}$ iff $C^\mathcal{I} = D^\mathcal{I}$
- $C(a)$ is true in $\mathcal{I}$ iff $a^\mathcal{I} \in C^\mathcal{I}$
- $R(a, b)$ is true in $\mathcal{I}$ iff $(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$
Reasoning in $\mathcal{ALC}$

- Sentences: Axioms or assertions
- $\mathcal{I}$ is a *model* for a sentence $S$ iff $S$ is true in $\mathcal{I}$
- $\mathcal{I}$ is a model for a DL knowledge base $K$ iff it is a model for every sentence in $K$
- Models of $K$ are denoted by $[K]$
- $S$ is *entailed* by $K$, written $K \models S$ iff $[K] \subseteq [S]$ (i.e. every model of $K$ is a model of $S$.)


Types of Reasoning in $\mathcal{ALC}$

$K$ a DL knowledge base;
$C$ and $D$ are concepts;
$R$ is a role;
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- Instance checking: $K \models C(a)$ or $K \models R(a, b)$
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- Concept satisfiability: $K \not\models C \sqsubseteq \bot$
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Reduction to Consistency Checking

Let $b$ be a new individual

- **Instance checking:**
  \[ K \models C(a) \text{ iff } K \cup \{ \neg C(a) \} \models \top \sqsubseteq \bot \]

- **Subsumption checking:**
  \[ K \models C \sqsubseteq D \text{ iff } K \cup \{ (C \sqcap \neg D)(b) \} \models \top \sqsubseteq \bot \]

- **Equivalence checking:**
  \[ K \models C \equiv D \text{ iff } K \cup \{ (C \sqcap \neg D)(b), (\neg C \sqcap D)(b) \} \models \top \sqsubseteq \bot \]

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Aside: Extensions to \( ALC \)

- There are many other possible constructors that can be added

- Extended concepts
  - Number restrictions: \((\leq n_R, C)\) and \((\geq n_R, C)\)

- Nominals: Allow individuals in the TBox

- Role operators
  - Inverse roles: \(R^{-}\) where \(R\) is a role

- Role axioms
  - Role hierarchy: \(R \sqsubseteq S\) where \(R\) and \(S\) are roles
    - So far have just used \(\sqsubseteq\) for concepts.
  - Transitive roles: \(R \in R^+\) where \(R\) is a role
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So far have just used $\sqsubseteq$ for concepts.
Extensions to $\mathcal{ALC}$: Examples

- $\text{ParentWithManySons} \triangleq (\geq 3\text{ParentOf}.\text{Male})$
- $\text{IndianCitizen} \triangleq \text{Person} \sqcap \exists \text{CitizenOf}.\{\text{India}\}$
- $\exists \text{ParentOf}^- \text{.Citizen} \sqsubseteq \text{Citizen}$
- $\text{ParentOf} \sqsubseteq \text{AncestorOf}$
- $\text{AncestorOf} \in R^+$
Extensions to $\mathcal{ALC}$: Semantics

- $(\leq nR.C)^I = \{x \mid |\{y \in C^I \mid (x, y) \in R^I\}| \leq n\}$
- $(\geq nR.C)^I = \{x \mid |\{y \in C^I \mid (x, y) \in R^I\}| \geq n\}$
- Inverse roles: $(R^-)^I = \{(y, x) \mid (x, y) \in R^I\}$
- $R \sqsubseteq S$ is true in $I$ iff $R^I \subseteq S^I$ for roles $R$ and $S$.
- $R \in R^+$ is true in $I$ iff
  $$(x, z) \in R^I \text{ whenever } (x, y) \in R^I \text{ and } (y, z) \in R^I$$
A Tableau Algorithm for \textit{ALC}

Goal: Show $KB \models A \sqsubseteq B$ by showing $KB \cup \{A \sqcap \neg B\}$ unsatisfiable.

Assume an \textit{unfoldable terminology}:

- Axioms are of the form $A \sqsubseteq C$ and $A \equiv C$ where $A$ is a concept name.
- For each concept name $A$, at most one axiom of the form $A \sqsubseteq C$ or $A \equiv C$.
- Axioms are acyclic:
  - $A \sqsubseteq C$ or $A \equiv C$ \textit{directly uses} a concept name $A_1$ iff $A_1$ occurs in $C$.
  - $A \sqsubseteq C$ or $A \equiv C$ \textit{uses} a concept name $A_1$ iff it directly uses $A_1$ or it directly uses a concept name $A_2$ and $A_2$ uses $A_1$.
  - $A \sqsubseteq C$ or $A \equiv C$ is \textit{acyclic} iff it does not use $A$. 
General Method

Show $KB \models A \subseteq B$ by showing $KB \cup \{A \cap \neg B\}$ is unsatisfiable.

Try to prove concept (un)satisfiability by constructing a model.

- A *tableau* is a graph representing such a model.
- A set of tableau *expansion rules* is used to construct the tableau.
- Either a model is constructed or a contradiction is found.
General Method

At the start:

- Assume an unfoldable terminology.
- Assume that all axioms are of the form $P \models Q$
  - This can be done by replacing any axiom of the form $A \sqsubseteq B$ by $A \models B \cap C$ where $C$ is a new concept name.
General Method

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- Assume that all axioms are of the form $P \Downarrow Q$
  - This can be done by replacing any axiom of the form $A \sqsubseteq B$ by $A \Downarrow B \sqcap C$ where $C$ is a new concept name.

If the query is $A \sqsubseteq B$:

- *negate* the query to get $A \sqcap \neg B$ (to show unsatisfiable);
- *unfold* the negated query;
- *convert* to *negation normal form*. 
General Method

At the start:

- Assume an unfoldable terminology.
- Assume that all axioms are of the form $P \vdash Q$
  - This can be done by replacing any axiom of the form $A \sqsubseteq B$ by $A \sqsupseteq B \cap C$ where $C$ is a new concept name.

If the query is $A \sqsubseteq B$:

- **negate** the query to get $A \sqcap \neg B$ (to show unsatisfiable);
- **unfold** the negated query;
- **convert** to negation normal form.

⚠️ Once the negated query has been unfolded, the rest of the KB can be ignored.
Unfold:
Expand every concept name occurring in the (negated) query.

- I.e. if concept $C$ appears in the query and $C \sqsubseteq D$ is in the KB, replace $C$ by $D$ in the query.
- Recall that for $C \sqsubseteq D$ in the KB, $C$ is a concept name and $D$ is an arbitrary $\mathcal{ALC}$ concept expression.
To Start

Unfold:
Expand every concept name occurring in the (negated) query.
  • I.e. if concept $C$ appears in the query and $C \sqsubseteq D$ is in the KB, replace $C$ by $D$ in the query.
    • Recall that for $C \sqsubseteq D$ in the KB, $C$ is a concept name and $D$ is an arbitrary $\mathcal{ALC}$ concept expression.

Negation normal form:
Negation occurs only in front of concept names
  • $\neg (C \sqcap D)$ gives $\neg C \sqcup \neg D$, and
  • $\neg (C \sqcup D)$ gives $\neg C \sqcap \neg D$
  • $\neg \exists R.C$ gives $\forall R.\neg C$, and
  • $\neg \forall R.C$ gives $\exists R.\neg C$
  • $\neg \neg C$ gives $C$
Algorithm

- Use a tree to represent the model being constructed
- Each node $x$ represents an individual, labelled with a set $L(x)$ of concepts it has to satisfy
  - $C \in L(x)$ implies $x \in C^I$
- Each edge $(x, y)$ represents a pair occurring in the interpretation of a role, labelled with the role name
  - $R = L((x, y))$ implies $(x, y) \in R^I$
To Determine the Satisfiability of a Concept C

- Initialise the tree $T$ with a single node $x$ with $L(x) = \{C\}$.
- Expand by repeatedly applying a set of **expansion rules**.
- $T$ is **fully expanded** when none of the rules can be applied.
- $T$ contains a **clash** when, for a node $y$ and a concept $D$,
  $\bot \in L(y)$ or $\{D, \neg D\} \subseteq L(y)$.
- If $T$ can’t be expanded without producing a clash, the concept is unsatisfiable.
Expansion Rules

(∩-rule) If \((C_1 \cap C_2) \in L(x)\) and \(\{C_1, C_2\} \not\subseteq L(x)\) then:
Add \(C_1\) and \(C_2\) to \(L(x)\).
Expansion Rules

(\cap\text{-rule}) If \((C_1 \cap C_2) \in L(x)\) and \(\{C_1, C_2\} \not\subseteq L(x)\) then:
Add \(C_1\) and \(C_2\) to \(L(x)\).

(\cup\text{-rule}) If \((C_1 \cup C_2) \in L(x)\) and \(\{C_1, C_2\} \cap L(x) = \emptyset\) then:
Add \(C_1\) to \(L(x)\).
If this leads to a clash, go back and add \(C_2\) to \(L(x)\).
Expansion Rules

(\cap\text{-rule}) If \((C_1 \cap C_2) \in L(x)\) and \(\{C_1, C_2\} \not\subseteq L(x)\) then:
Add \(C_1\) and \(C_2\) to \(L(x)\).

(\cup\text{-rule}) If \((C_1 \cup C_2) \in L(x)\) and \(\{C_1, C_2\} \cap L(x) = \emptyset\) then:
Add \(C_1\) to \(L(x)\).
If this leads to a clash, go back and add \(C_2\) to \(L(x)\).

(\exists\text{-rule}) If \(\exists R. C \in L(x)\) and there is no \(y\) s.t. \(L((x, y)) = R\) and \(C \in L(y)\) then:
Create a new node \(y\) and edge \((x, y)\) with \(L(y) = C\) and \(L((x, y)) = R\).
Expansion Rules

(\cap\text{-rule}) If \((C_1 \cap C_2) \in L(x)\) and \(\{C_1, C_2\} \not\subseteq L(x)\) then:
Add \(C_1\) and \(C_2\) to \(L(x)\).

(\cup\text{-rule}) If \((C_1 \cup C_2) \in L(x)\) and \(\{C_1, C_2\} \cap L(x) = \emptyset\) then:
Add \(C_1\) to \(L(x)\).
If this leads to a clash, go back and add \(C_2\) to \(L(x)\).

(\exists\text{-rule}) If \(\exists R.C \in L(x)\) and there is no \(y\) s.t. \(L((x, y)) = R\) and \(C \in L(y)\) then:
Create a new node \(y\) and edge \((x, y)\) with \(L(y) = C\) and \(L((x, y)) = R\).

(\forall\text{-rule}) If \(\forall R.C \in L(x)\) and there is some \(y\) s.t. \(L((x, y)) = R\) and \(C \not\in L(y)\) then:
Add \(C\) to \(L(y)\).
Interpreting a tree $T$

- If $T$ contains a clash the concept $C$ is unsatisfiable.
- If $T$ is fully expanded and clash-free, then $C$ is satisfiable.
- In the second case, construct a model $I$ as follows:
  - $\Delta = \{x \mid x \text{ is a node in } T\}$.
  - $A^I = \{x \in \Delta \mid A \in L(x)\}$ for all concept names $A$ in $C$.
  - $R^I = \{(x, y) \mid (x, y) \text{ is an edge in } T \text{ and } L((x, y)) = R\}$.
Termination of the Algorithm

- The \( \cap \)-, \( \cup \)-and \( \exists \)-rules can only be applied once to a concept in \( L(x) \).
- The \( \forall \)-rule can be applied many times to a given \( \forall R.C \) expression in \( L(x) \), but only once to a given edge \((x, y)\).
- Applying any rule to a concept \( C \) extends the labelling with a concept strictly smaller than \( C \).

Therefore the algorithm must terminate.
Tableau Algorithm: Example 1

DL knowledge base:

- \( \text{vegan} \equiv \text{person} \sqcap \forall \text{eats.plant} \)
- \( \text{vegetarian} \equiv \text{person} \sqcap \forall \text{eats.(plants \sqcup dairy)} \)

Query: \( \text{vegan} \sqsubseteq \text{vegetarian} \)

Convert to:

- \( \text{vegan} \sqcap \neg \text{vegetarian} \text{ is unsatisfiable?} \)
Example 1

- Unfold and normalise $\text{vegan} \sqcap \neg \text{vegetarian}$:
  
  $\text{person} \sqcap \forall \text{eats}. \text{plant} \sqcap (\neg \text{person} \sqcup \exists \text{eats}.(\neg \text{plant} \sqcap \neg \text{dairy}))$
Example 1

- Unfold and normalise \( \text{vegan} \sqcap \neg \text{vegetarian} \):
  \[ \text{person} \sqcap \forall \text{eats}.\text{plant} \sqcap (\neg \text{person} \sqcup \exists \text{eats}.(\neg \text{plant} \sqcap \neg \text{dairy})) \]

- Initialise \( T \) to \( L(x) \) to contain:
  \[ \text{person} \sqcap \forall \text{eats}.\text{plant} \sqcap (\neg \text{person} \sqcup \exists \text{eats}.(\neg \text{plant} \sqcap \neg \text{dairy})) \]
Example 1

- Unfold and normalise `vegan ⊓ ¬vegetarian`:
  \[ \text{person} ⊓ \forall \text{eats.plant} ⊓ (¬\text{person} ⊔ ∃ \text{eats.} (¬\text{plant} ⊓ ¬\text{dairy})) \]

- Initialise \( T \) to \( L(x) \) to contain:
  \[ \text{person} ⊓ \forall \text{eats.plant} ⊓ (¬\text{person} ⊔ ∃ \text{eats.} (¬\text{plant} ⊓ ¬\text{dairy})) \]

- Apply \( ∩ \)-rule and add to \( L(x) \):
  \[ \{ \text{person}, \forall \text{eats.plant}, ¬\text{person} ⊔ ∃ \text{eats.} (¬\text{plant} ⊓ ¬\text{dairy}) \} \]
Example 1

- Apply $\sqcup$-rule to $\neg person \sqcup \exists eats.(\neg plant \land \neg dairy)$:
  - Add $\neg person$ to $L(x)$: Clash
  - Go back and add $\exists eats. (\neg plant \land \neg dairy)$ to $L(x)$
Example 1

- Apply $\sqcup$-rule to $\neg$person $\sqcup \exists$eats.$(\neg$plant $\sqcap \neg$dairy$):$
  Add $\neg$person to $L(x)$: Clash
  Go back and add $\exists$eats.$(\neg$plant $\sqcap \neg$dairy$)$ to $L(x)$

- Apply $\exists$-rule to $\exists$eats.$(\neg$plant $\sqcap \neg$dairy$)$:
  Create new node $y$ and new edge $(x, y)$:
  $L(y) = \{\neg$plant $\sqcap \neg$dairy$\}; L((x, y)) = eats$
Example 1

- Apply $\sqcup$-rule to $\neg person \sqcup \exists eats. (\neg plant \land \neg dairy)$:
  - Add $\neg person$ to $L(x)$: Clash
  - Go back and add $\exists eats. (\neg plant \land \neg dairy)$ to $L(x)$

- Apply $\exists$-rule to $\exists eats. (\neg plant \land \neg dairy)$:
  - Create new node $y$ and new edge $(x, y)$:
    
    $L(y) = \{\neg plant \land \neg dairy\}; \quad L((x, y)) = eats$

- Apply $\forall$-rule to $\forall eats.plant$ in $L(x)$ and $L((x, y)) = eats$:
  - Add $plant$ to $L(y)$
Example 1

- Apply \( \sqcap \)-rule to \( \neg plant \sqcap \neg dairy \) in \( L(y) \):
  Add \( \{\neg plant, \neg dairy\} \) to \( L(y) \): Clash

Conclusion

- Both applications of the \( \sqcup \)-rule lead to clashes
- So vegan \( \sqcap \neg vegetarian \) is unsatisfiable
- So vegan \( \sqsubseteq vegetarian \)
Example 1

• Apply $\neg$-rule to $\neg plant \sqcap \neg dairy$ in $L(y)$:
  Add $\{\neg plant, \neg dairy\}$ to $L(y)$: Clash

• Conclusion
  • Both applications of the $\sqcup$-rule lead to clashes
  • So $\text{vegan} \sqcap \neg \text{vegetarian}$ is unsatisfiable
  • So $\text{vegan} \sqsubseteq \text{vegetarian}$
Example 2

- Query: \( \text{vegetarian} \sqsubseteq \text{vegan} \)
- Convert to: \( \text{vegetarian} \sqcap \neg \text{vegan} \) is satisfiable?
- Unfold and normalise \( \text{vegetarian} \sqcap \neg \text{vegan} \):
  \[
  \text{person} \sqcap \forall \text{eats.} (\text{plant} \sqcup \text{dairy}) \sqcap (\neg \text{person} \sqcup \exists \text{eats.} \neg \text{plant})
  \]
- Initialise \( T \) to \( L(x) \) to contain:
  \[
  \{ \text{person} \sqcap \forall \text{eats.} (\text{plant} \sqcup \text{dairy}) \sqcap (\neg \text{person} \sqcup \exists \text{eats.} \neg \text{plant}) \}
  \]
Example 2

- Apply \( \sqcap \)-rule and add to \( L(x) \):
  \[
  \{ \text{person}, \forall \text{eats.} (\text{plant} \sqcup \text{dairy}), \neg \text{person} \sqcup \exists \text{eats.} \neg \text{plant} \}
  \]
Example 2

• Apply $\Box$-rule and add to $L(x)$:
  \[
  \{\text{person}, \forall \text{eats.}(\text{plant} \sqcup \text{dairy}), \neg \text{person} \sqcup \exists \text{eats.}\neg \text{plant}\}\]

• Apply $\sqcup$-rule to $\neg \text{person} \sqcup \exists \text{eats.}\neg \text{plant}$:
  Add $\neg \text{person}$ to $L(x)$: Clash
  Go back and add $\exists \text{eats.}\neg \text{plant}$ to $L(x)$
Example 2

• Apply $\sqcap$-rule and add to $L(x)$:
  \{\textit{person}, $\forall$\textit{eats.\,(\textit{plant} $\sqcup$ \textit{dairy})}, $\neg$\textit{person} $\sqcup$ $\exists$\textit{eats.\,$\neg$\textit{plant}}\}

• Apply $\sqcup$-rule to $\neg$\textit{person} $\sqcup$ $\exists$\textit{eats.\,$\neg$\textit{plant}}:
  Add $\neg$\textit{person} to $L(x)$: Clash
  Go back and add $\exists$\textit{eats.\,$\neg$\textit{plant}} to $L(x)$

• Apply $\exists$-rule to $\exists$\textit{eats.\,$\neg$\textit{plant}}:
  Create new node $y$ and new edge $(x, y)$
  $L(y) = \{\neg$\textit{plant}\}; L((x, y)) = \textit{eats}$
Example 2

- Apply $\forall$-rule to $\forall eats. (plant \sqcup dairy)$ in $L(x)$ and $L((x, y)) = eats$:
  Add $plant \sqcup dairy$ to $L(y)$
Example 2

- Apply $\forall$-rule to $\forall eats. (plant \sqcup dairy)$ in $L(x)$ and $L((x, y)) = eats$:
  Add $plant \sqcup dairy$ to $L(y)$

- Apply $\sqcup$-rule to $plant \sqcup dairy$ in $L(y)$:
  Add $plant$ to $L(y)$: Clash
  Go back and add $dairy$ to $L(y)$

- Conclusion
  - No rules are applicable, so $T$ is fully expanded
  - So vegetarian $\sqcap \neg$ vegan is satisfiable
  - So vegetarian $\not\sqsubseteq$ vegan
Example 2

- Apply $\forall$-rule to $\forall e (\text{plant} \sqcup \text{dairy})$ in $L(x)$ and $L((x, y)) = e$:
  Add $\text{plant} \sqcup \text{dairy}$ to $L(y)$

- Apply $\sqcup$-rule to $\text{plant} \sqcup \text{dairy}$ in $L(y)$:
  Add $\text{plant}$ to $L(y)$: Clash
  Go back and add $\text{dairy}$ to $L(y)$

- Conclusion
  - No rules are applicable, so $T$ is fully expanded
  - So $\text{vegetarian} \sqcap \neg \text{vegan}$ is satisfiable
  - So $\text{vegetarian} \not\sqsubseteq \text{vegan}$
The Brachman&Levesque DL and $\mathcal{ALC}$

<table>
<thead>
<tr>
<th>Constructor</th>
<th>B&amp;L</th>
<th>$\mathcal{ALC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conj.</td>
<td>$(\text{AND } A \  B)$</td>
<td>$A \sqcap B$</td>
</tr>
<tr>
<td>Univ. quant.</td>
<td>$(\text{ALL } R \ C)$</td>
<td>$\forall R. C$</td>
</tr>
<tr>
<td>Exist. quant.</td>
<td></td>
<td>$\exists R. C$</td>
</tr>
<tr>
<td>Unqual. exist. quant.</td>
<td>$(\text{EXISTS } 1 \ R)$</td>
<td>$\exists R. \top$</td>
</tr>
<tr>
<td>Number restriction</td>
<td>$(\text{EXISTS } n \ R)$</td>
<td></td>
</tr>
<tr>
<td>Role filler</td>
<td>$(\text{FILLS } R \ a)$</td>
<td></td>
</tr>
<tr>
<td>Assertion</td>
<td>$a \rightarrow C$</td>
<td>$C(a)$</td>
</tr>
</tbody>
</table>

- $\mathcal{FL}^-$ consists of Conj., Univ. quant., and Unqual. exist. quant.
- The B&L DL is slightly more general than $\mathcal{FL}^-$.
- $\mathcal{ALC}$ is $\mathcal{FL}^-$ plus $\top$, $\bot$, and general negation.
- The extension to $\mathcal{ALC}$ for a role filler would use $\forall R. \{a\}$. 
References

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