Description Logics: $\textit{ALC}$
Outline

Topics:

1. Introduction to description logics
2. The description logic $ALC$
3. Extensions to $ALC$
4. A tableau algorithm for $ALC$
### Introduction

**Description logics**

- A DL is a formalism for expressing *concepts*, their attributes (or associated *roles*), and the *relationships* between them.
  - E.g. *Person* could be a concept and a role could be *ParentOf*.
- Can be regarded as a KR system based on a *structured representation of knowledge*.
- Most DLs are fragments of FOL, written in a distinct syntax.

**Predecessors of DLs**

- Semantic networks of the 70s
- Frame-based systems
Why Description Logics?

Ideal AI case:

- Approaches have scientific (logical) and engineering aspects
- **Scientific**: Analyse the problem formally and in detail
- **Engineering**: Get something working quickly and efficiently
- Success: *When these two approaches coincide – efficient implementations of (formally) well-understood systems.*

- Description Logic research has (arguably) reached this point
Background: Concepts, Roles, Constants

- In a description logic, there are sentences that will be true or false (as in FOL).
  - These are restricted to *subsumption* and *instance* assertions.
- In addition, there are three sorts of expressions that act like nouns and noun phrases in English:
  - **Concepts** are like category nouns: Person, Female, GraduateStudent
  - **Roles** are like relational nouns: AgeOf, ParentOf, AreaOfStudy
    - Specify attributes of concepts and their types
  - **Constants** are like proper nouns: John, Mary
- These correspond to unary predicates, binary predicates and constants (respectively) in FOL.
- Unlike in FOL, concepts need not be atomic and can have structure.
An KB in a DL contains two parts:

- Define terminology:  \textit{TBox}
  - E.g.  \textit{MWD} \models \text{Mother} \sqcap \forall \text{ParentOf} . \neg \text{Female}

- Give assertions:  \textit{ABox}
  - E.g.  \textit{MWD}(\text{sue}).
Main components of the TBox:

- **Concepts**: classes of individuals
  - E.g. *Mother*
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- **Complex concepts** using constructors
  - E.g. $\text{Mother} \sqcap \forall\text{ParentOf}. \neg\text{Female}$

- **Assertions** concerning complex concepts
  - E.g. $\text{MWD} \models \text{Mother} \sqcap \forall\text{ParentOf}. \neg\text{Female}$
  - $\text{Mother} \sqsubseteq \text{Female}$
DL Knowledge Bases: ABox

ABox: Assertions that individuals satisfy certain concepts and roles.

- Think of as a simple relational database.
- E.g. $MWD(Mary)$, $ParentOf(Mary, John)$. 
DL: Advantages

- Well-defined formal semantics.
- Known (and often good) complexity characteristics or implementations.
- Relatively easy to specify DL knowledge bases, in a structured hierarchical fashion.
- DLs constitute a large family of approaches.
  - Can tailor a language to a specific application.
Applications

Useful whenever a common vocabulary is important.

E.g.:
- Enhanced database systems
  - DL-Lite
- Medical informatics: Snomed CT, Galen
  - EL
- Semantic Web
  - Next generation web
  - OWL: W3C recommendation.

We’ll look at perhaps the most central DL, $\mathcal{ALC}$. 
An $\mathcal{ALC}$ KB contains two parts:

- Define terminology: TBox
- Give assertions: ABox
The Logic $\textit{ALC}$

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- Define terminology: TBox
- Give assertions: ABox

Main components of the TBox:

- Concepts: Represent classes of individuals
- Roles: Represent binary relations between individuals
- Complex concepts using constructors

Examples:

- Concept names: Person, Female
- Role names: ParentOf, HasHusband
- Individual names (in the ABox): John, Mary
The Logic $\mathcal{ALC}$: Language

Logical symbols:

- Propositional constructors: $\sqcap$, $\sqcup$, $\neg$
- Other restrictions: $\forall$, $\exists$
  - Note: These are different from quantifiers as seen in FOL
- $\top$, $\bot$
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Concept construction
- Let $C$ and $D$ be concepts and $R$ a role.
- $\neg C$, $C \sqcap D$, $C \sqcup D$ are concepts.
- $\forall R.C$, $\exists R.C$ are concepts.
The Logic $\mathcal{ALC}$: Language

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Let $C$ and $D$ be concepts and $R$ a role.

- $C$ stands for a concept or set of individuals.
- $\neg C$ stands for the concept of things that are not a $C$.
- $C \sqcap D$ is the concept of things that are both $C$ and $D$.
- $\text{E.g. } \text{Female } \sqcap \text{Human}$
- $C \sqcup D$ is the concept of things that are either $C$ or $D$ or both.
- $\text{E.g. } \text{Male } \sqcup \text{Female}$
- $\forall R.C$ is the concept of things such that all things that are $R$ related to it are $C$'s.
- $\text{E.g. } \forall \text{ParentOf}. \text{Female}$: things all of whose children are female
- $\exists R.C$ is the concept of things such that some thing $R$ related to it is a $C$.
- $\exists \text{ParentOf}. \text{Female}$: things with a female child
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The Logic $\mathcal{ALC}$: Knowledge Bases

Axioms (assertions) in the TBox:

- Subsumption: $C \sqsubseteq D$ where $C$ and $D$ are concepts
- Equivalence axioms: $C \equiv D$ where $C$ and $D$ are concepts
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Assertions in the ABox:

- $C(a)$ where $C$ is a concept and $a$ is an individual name.
- $R(a, b)$ where $R$ is a role name, $a$ and $b$ are individual names.
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DL knowledge base:

- Set of TBox statements
- Set of ABox statements
Examples

TBox:

- \textit{Person} \sqsubseteq \textit{Animal} \sqcap \textit{Biped}
- \textit{Woman} \models \textit{Person} \sqcap \textit{Female}
- \textit{Mother} \models \textit{Woman} \sqcap \exists \text{ParentOf}. \textit{Person}
- \textit{Parent} \models \textit{Mother} \sqcup \textit{Father}
- \textit{Man} \models \textit{Person} \sqcap \neg \textit{Woman}
- \textit{MotherWithoutDaughter} \models \textit{Mother} \sqcap \forall \text{ParentOf}. \neg \textit{Female}
- \textit{GrandMother} \models \textit{Woman} \sqcap \exists \text{ParentOf}. \textit{Parent}

ABox:

- \textit{GrandMother}(\textit{Sally})
- (\textit{Person} \sqcap \textit{Male})(\textit{John})
Formal Semantics for Concepts and Names

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An interpretation is a pair $\mathcal{I} = \langle \Delta, \mathcal{I} \rangle$

- Domain $\Delta$: non-empty set of objects
- Interpretation function $\mathcal{I}$: Maps structures into the domain.
- Recall, Brachman and Levesque write this as $\mathcal{I} = \langle D, I \rangle$. 
Formal Semantics for Concepts and Names

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Then:

- $^\mathcal{I}$ maps every concept name $A$ to a subset $A^\mathcal{I} \subseteq \Delta$
- $^\mathcal{I}$ maps every role name $R$ to a binary relation $R^\mathcal{I} \subseteq \Delta \times \Delta$
- $^\mathcal{I}$ maps individual names $a$ to elements of $\Delta : a^\mathcal{I} \in \Delta$
- $\top^\mathcal{I} = \Delta$ and $\bot^\mathcal{I} = \emptyset$. 
Semantics for Complex Concepts

Assume $C$, $D$ are concepts, and $R$ is a role.

- $(\neg C)^I = \Delta \setminus C^I$
- $(C \cap D)^I = C^I \cap D^I$
- $(C \cup D)^I = C^I \cup D^I$
- $(\forall R. C)^I = \{ x \mid y \in C^I \text{ for every } y \text{ s.t. } (x, y) \in R^I \}$
- $(\exists R. C)^I = \{ x \mid y \in C^I \text{ for some } y \text{ s.t. } (x, y) \in R^I \}$
Semantics for Axioms and Assertions

Assume $C$, $D$ are concepts, $R$ is a role, $a$ and $b$ are individual names. Let $\mathcal{I} = (\Delta, ^{\mathcal{I}})$ be an interpretation.

- $C \sqsubseteq D$ is true in $\mathcal{I}$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$
- $C \sqsupset D$ is true in $\mathcal{I}$ iff $C^\mathcal{I} = D^\mathcal{I}$
- $C(a)$ is true in $\mathcal{I}$ iff $a^\mathcal{I} \in C^\mathcal{I}$
- $R(a, b)$ is true in $\mathcal{I}$ iff $(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$
Reasoning in \textit{ALC}

- Sentences: Axioms or assertions
- $\mathcal{I}$ is a \textit{model} for a sentence $S$ iff $S$ is true in $\mathcal{I}$
- $\mathcal{I}$ is a model for a DL knowledge base $K$ iff it is a model for every sentence in $K$
- Models of $K$ are denoted by $[K]$
- $S$ is \textit{entailed} by $K$, written $K \models S$ iff $[K] \subseteq [S]$ (i.e. every model of $K$ is a model of $S$.)
Types of Reasoning in $\mathcal{ALC}$

$K$ a DL knowledge base;
$C$ and $D$ are concepts;
$R$ is a role;
a and $b$ are individual names

- Instance checking: $K \models C(a)$ or $K \models R(a, b)$
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Reduction to Consistency Checking

Let $b$ be a new individual

- Instance checking:
  $K ⊨ C(a)$ iff $K \cup \{\neg C(a)\} \models \top \sqsubseteq \bot$

- Subsumption checking:
  $K \models C \sqsubseteq D$ iff $K \cup \{C \cap \neg D(b)\} \models \top \sqsubseteq \bot$

- Equivalence checking:
  $K \models C . = D$ iff $K \cup \{C \cap \neg D(b), \neg C \cap D(b)\} \models \top \sqsubseteq \bot$

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- **Subsumption checking:**
  \[ K \models C \sqsubseteq D \iff K \cup \{((C \sqcap \neg D)(b))\} \models \top \sqsubseteq \bot \]

- **Equivalence checking:**
  \[ K \models C = D \iff K \cup \{((C \sqcap \neg D)(b)), ((\neg C \sqcap D)(b))\} \models \top \sqsubseteq \bot \]

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Aside: Extensions to $\mathcal{ALC}$

There are many other possible constructors that can be added

- Extended concepts
  - Number restrictions: $(\leq n_R. C)$ and $(\geq n_R. C)$
    - E.g. $\text{ParentWithManySons} = (\geq 3 \text{ParentOf}. \text{Male})$
  - Nominals: Allow individuals in the TBox
    - E.g. $\text{IndianCitizen} = \text{Person} \sqcap \exists \text{CitizenOf}. \{\text{India}\}$

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Extensions to \( \mathcal{ALC} \)

Role operators

- Inverse roles: \( R^\sim \) where \( R \) is a role
Extensions to $\mathcal{ALC}$

Role operators

- Inverse roles: $R^-$ where $R$ is a role
  - E.g. $\exists ParentOf^- . Citizen \sqsubseteq Citizen$
  - $GradCourse \sqsubseteq \forall teaches^- . Professor$
Extensions to $\mathcal{ALC}$

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  E.g. $\exists ParentOf^-.Citizen \sqsubseteq Citizen$

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Role axioms

- Role hierarchy: $R \sqsubseteq S$ where $R$ and $S$ are roles

  \[ \text{So far have just used } \sqsubseteq \text{ for concepts.} \]
Extensions to ALC

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- Transitive roles: $R \in R^+$ where $R$ is a role
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  E.g. $AncestorOf \in R^+$

And lots of others . . .
Extensions to $\mathcal{ALC}$: Semantics

- $(\leq nR.C)^{I} = \{ x \mid |\{y \in C^{I} \mid (x,y) \in R^{I}\}| \leq n\}$
- $(\geq nR.C)^{I} = \{ x \mid |\{y \in C^{I} \mid (x,y) \in R^{I}\}| \geq n\}$
- Inverse roles: $(R^{-})^{I} = \{ (y,x) \mid (x,y) \in R^{I}\}$
- $R \sqsubseteq S$ is true in $I$ iff $R^{I} \subseteq S^{I}$ for roles $R$ and $S$.
- $R \in R^{+}$ is true in $I$ iff $(x,z) \in R^{I}$ whenever $(x,y) \in R^{I}$ and $(y,z) \in R^{I}$
A Tableau Algorithm for $\mathcal{ALC}$

Goal: Show $KB \models A \sqsubseteq B$ by showing $KB \cup \{A \sqcap \neg B\}$ unsatisfiable.
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Assume an *unfoldable terminology*: 

• Axioms are of the form $A \sqsubseteq C$ and $A = C$ where $A$ is a concept name.

• For each concept name $A$, at most one axiom of the form $A \sqsubseteq C$ or $A = C$.

• Axioms are acyclic:

  • $A \sqsubseteq C$ or $A = C$ directly uses a concept name $A_1$ iff $A_1$ occurs in $C$.

  • $A \sqsubseteq C$ or $A = C$ uses a concept name $A_1$ iff it directly uses $A_1$ or it directly uses a concept name $A_2$ and $A_2$ uses $A_1$.

  • $A \sqsubseteq C$ or $A = C$ is acyclic iff it does not use $A$. 
A Tableau Algorithm for $\mathcal{ALC}$

**Goal:** Show $KB \models A \sqsubseteq B$ by showing $KB \cup \{A \sqcap \neg B\}$ unsatisfiable.

Assume an *unfoldable terminology*:

- Axioms are of the form $A \sqsubseteq C$ and $A \doteq C$ where $A$ is a concept name.
A Tableau Algorithm for $\mathcal{ALC}$

Goal: Show $KB \models A \sqsubseteq B$ by showing $KB \cup \{A \sqcap \neg B\}$ unsatisfiable.

Assume an *unfoldable terminology*:

- Axioms are of the form $A \sqsubseteq C$ and $A \dot{=} C$ where $A$ is a concept name.
- For each concept name $A$, at most one axiom of the form $A \sqsubseteq C$ or $A \dot{=} C$. 
A Tableau Algorithm for $\mathcal{ALC}$

Goal: Show $KB \models A \sqsubseteq B$ by showing $KB \cup \{A \sqcap \neg B\}$ unsatisfiable.

Assume an *unfoldable terminology*:

- Axioms are of the form $A \sqsubseteq C$ and $A \equiv C$ where $A$ is a concept name.
- For each concept name $A$, at most one axiom of the form $A \sqsubseteq C$ or $A \equiv C$.
- Axioms are acyclic:
  - $A \sqsubseteq C$ or $A \equiv C$ *directly uses* a concept name $A_1$ iff $A_1$ occurs in $C$.
  - $A \sqsubseteq C$ or $A \equiv C$ *uses* a concept name $A_1$ iff it directly uses $A_1$ or it directly uses a concept name $A_2$ and $A_2$ uses $A_1$.
  - $A \sqsubseteq C$ or $A \equiv C$ is *acyclic* iff it does not use $A$. 
General Method

Show $KB \models A \subseteq B$ by showing $KB \cup \{A \cap \neg B\}$ is unsatisfiable.

Try to prove concept (un)satisfiability by constructing a model.

- A **tableau** is a graph representing such a model.
- A set of tableau **expansion rules** is used to construct the tableau.
- Either a model is constructed or a contradiction is found.
General Method

At the start:

- Assume an unfoldable terminology.
- Assume that all axioms are of the form $P \sqsubseteq Q$
  - This can be done by replacing any axiom of the form $A \sqsubseteq B$ by $A \sqsubseteq B \sqcap C$ where $C$ is a new concept name.
General Method

At the start:

- Assume an unfoldable terminology.
- Assume that all axioms are of the form $P \vdash Q$
  - This can be done by replacing any axiom of the form $A \sqsubseteq B$ by $A \vdash B \sqcap C$ where $C$ is a new concept name.

If the query is $A \sqsubseteq B$, first convert to a normal form:

- **negate** the query to get $A \sqcap \neg B$ (to show unsatisfiable);
- **unfold** the negated query;
- **convert** to *negation normal form*. 
General Method

At the start:

- Assume an unfoldable terminology.
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  - This can be done by replacing any axiom of the form $A \sqsubseteq B$ by $A \equiv B \cap C$ where $C$ is a new concept name.

If the query is $A \sqsubseteq B$, first convert to a normal form:

- *negate* the query to get $A \sqcap \neg B$ (to show unsatisfiable);
- *unfold* the negated query;
- *convert* to *negation normal form*.

Once the negated query has been unfolded, the rest of the KB can be ignored.
To Unfold:
Expand every concept name occurring in the (negated) query.

- I.e. if concept $C$ appears in the query and $C \models D$ is in the KB, replace $C$ by $D$ in the query.

- Recall that for $C \models D$ in the KB, $C$ is a concept name and $D$ is an arbitrary $\mathcal{ALC}$ concept expression.

- As well, $C$ is guaranteed to not appear in $D$ or in any later substitutions.
Negation normal form

Negation normal form:
Move negation in so that it occurs only in front of concept names

- \( \neg(C \sqcap D) \) gives \( \neg C \sqcup \neg D \), and
  \( \neg(C \sqcup D) \) gives \( \neg C \sqcap \neg D \)
- \( \neg \exists R. C \) gives \( \forall R. \neg C \), and
  \( \neg \forall R. C \) gives \( \exists R. \neg C \)
- \( \neg \neg C \) gives \( C \)
Algorithm

- Use a tree to represent the model being constructed
- Each node $x$ represents an individual, labelled with a set $L(x)$ of concepts it has to satisfy
  - $C \in L(x)$ implies $x \in C^I$
- Each edge $(x, y)$ represents a pair occurring in the interpretation of a role, labelled with the role name
  - $R = L((x, y))$ implies $(x, y) \in R^I$
To Determine the Satisfiability of a Concept C

- Initialise the tree $T$ with a single node $x$ with $L(x) = \{C\}$.
- Expand by repeatedly applying a set of expansion rules.
- $T$ is fully expanded when none of the rules can be applied.
- $T$ contains a clash when, for a node $y$ and a concept $D$, $ot \in L(y)$ or $\{D, \neg D\} \subseteq L(y)$.
- If $T$ can’t be expanded without producing a clash, the concept is unsatisfiable.
Expansion Rules

(\ominus\text{-rule}) If \((C_1 \cap C_2) \in L(x)\) and \(\{C_1, C_2\} \not\subseteq L(x)\) then:
Add \(C_1\) and \(C_2\) to \(L(x)\).
Expansion Rules

(⊓-rule) If \((C_1 \cap C_2) \in L(x)\) and \(\{C_1, C_2\} \not\subseteq L(x)\) then:
Add \(C_1\) and \(C_2\) to \(L(x)\).

(⊔-rule) If \((C_1 \cup C_2) \in L(x)\) and \(\{C_1, C_2\} \cap L(x) = \emptyset\) then:
Add \(C_1\) to \(L(x)\).
If this leads to a clash, go back and add \(C_2\) to \(L(x)\).
Expansion Rules

(\text{-rule}) If \((C_1 \cap C_2) \in L(x)\) and \(\{C_1, C_2\} \not\subseteq L(x)\) then:
Add \(C_1\) and \(C_2\) to \(L(x)\).

(\text{-rule}) If \((C_1 \cup C_2) \in L(x)\) and \(\{C_1, C_2\} \cap L(x) = \emptyset\) then:
Add \(C_1\) to \(L(x)\).
If this leads to a clash, go back and add \(C_2\) to \(L(x)\).

(\exists\text{-rule}) If \(\exists R. C \in L(x)\) and there is no \(y\) s.t. \(L((x, y)) = R\) and \(C \in L(y)\) then:
Create a new node \(y\) and edge \((x, y)\) with \(L(y) = C\) and \(L((x, y)) = R\).
Expansion Rules

(\cap\text{-rule}) If \((C_1 \cap C_2) \in L(x)\) and \(\{C_1, C_2\} \not\subseteq L(x)\) then:
Add \(C_1\) and \(C_2\) to \(L(x)\).

(\cup\text{-rule}) If \((C_1 \cup C_2) \in L(x)\) and \(\{C_1, C_2\} \cap L(x) = \emptyset\) then:
Add \(C_1\) to \(L(x)\).
If this leads to a clash, go back and add \(C_2\) to \(L(x)\).

(\exists\text{-rule}) If \(\exists R. C \in L(x)\) and there is no \(y\) s.t. \(L((x, y)) = R\) and \(C \in L(y)\) then:
Create a new node \(y\) and edge \((x, y)\) with \(L(y) = C\) and \(L((x, y)) = R\).

(\forall\text{-rule}) If \(\forall R. C \in L(x)\) and there is some \(y\) s.t. \(L((x, y)) = R\) and \(C \not\in L(y)\) then:
Add \(C\) to \(L(y)\).
Interpreting a tree $T$

- If $T$ contains a clash the concept $C$ is unsatisfiable.
- If $T$ is fully expanded and clash-free, then $C$ is satisfiable.
- In the second case, construct a model $I$ as follows:
  - $\Delta = \{x \mid x \text{ is a node in } T\}$.
  - $A^I = \{x \in \Delta \mid A \in L(x)\}$ for all concept names $A$ in $C$.
  - $R^I = \{(x, y) \mid (x, y) \text{ is an edge in } T \text{ and } L((x, y)) = R\}$.
Termination of the Algorithm

- The $\forall$, $\exists$, and $\exists$-rules can only be applied once to a concept in $L(x)$.
- The $\forall$-rule can be applied many times to a given $\forall R.C$ expression in $L(x)$, but only once to a given edge $(x, y)$.
- Applying any rule to a concept $C$ extends the labelling with a concept strictly smaller than $C$.

Therefore the algorithm must terminate.
Tableau Algorithm: Example 1

DL knowledge base:

- $\text{vegan} \equiv \text{person} \sqcap \forall \text{eats}. \text{plant}$
- $\text{vegetarian} \equiv \text{person} \sqcap \forall \text{eats}. (\text{plants} \sqcup \text{dairy})$

Query: $\text{vegan} \sqsubseteq \text{vegetarian}$

Convert to:

- $\text{vegan} \sqcap \neg \text{vegetarian}$ is unsatisfiable?
Example 1

- Unfold and normalise $\text{vegan} \sqcap \neg \text{vegetarian}$:
  
  $\text{person} \sqcap \forall \text{eats} \cdot \text{plant} \sqcap (\neg \text{person} \sqcup \exists \text{eats} \cdot (\neg \text{plant} \sqcap \neg \text{dairy}))$
Example 1

- Unfold and normalise \( \text{vegan} \sqcap \neg \text{vegetarian} \):
  \[ \text{person} \sqcap \forall \text{eats}. \text{plant} \sqcap (\neg \text{person} \sqcup \exists \text{eats}.(\neg \text{plant} \sqcap \neg \text{dairy})) \]
- Initialise \( T \) to \( L(x) \) to contain:
  \[ \text{person} \sqcap \forall \text{eats}. \text{plant} \sqcap (\neg \text{person} \sqcup \exists \text{eats}.(\neg \text{plant} \sqcap \neg \text{dairy})) \]
Example 1

- Unfold and normalise $vegan \sqcap \neg vegetarian$:
  $person \sqcap \forall eats.\ plant \sqcap (\neg person \sqcup \exists eats. (\neg plant \sqcap \neg dairy))$

- Initialise $T$ to $L(x)$ to contain:
  $person \sqcap \forall eats.\ plant \sqcap (\neg person \sqcup \exists eats. (\neg plant \sqcap \neg dairy))$

- Apply $\sqcap$-rule and add to $L(x)$:
  $\{ person, \forall eats.\ plant, \neg person \sqcup \exists eats. (\neg plant \sqcap \neg dairy) \}$
Example 1

- Apply $\sqcup$-rule to \( \neg \text{person} \sqcup \exists \text{eats}.(\neg \text{plant} \sqcap \neg \text{dairy}) \):
  - Add \( \neg \text{person} \) to \( L(x) \): Clash
  - Go back and add \( \exists \text{eats}.(\neg \text{plant} \sqcap \neg \text{dairy}) \) to \( L(x) \)
Example 1

- Apply $\sqcup$-rule to $\neg\text{person} \sqcup \exists \text{eats.}(\neg \text{plant} \sqcap \neg \text{dairy})$:
  Add $\neg\text{person}$ to $L(x)$: Clash
  Go back and add $\exists \text{eats.}(\neg \text{plant} \sqcap \neg \text{dairy})$ to $L(x)$

- Apply $\exists$-rule to $\exists \text{eats.}(\neg \text{plant} \sqcap \neg \text{dairy})$:
  Create new node $y$ and new edge $(x, y)$:
  $L(y) = \{\neg \text{plant} \sqcap \neg \text{dairy}\}; L((x, y)) = \text{eats}$
Example 1

• Apply $\sqcup$-rule to $\neg person \sqcup \exists eats. (\neg plant \sqcap \neg dairy)$:
  Add $\neg person$ to $L(x)$: Clash
  Go back and add $\exists eats. (\neg plant \sqcap \neg dairy)$ to $L(x)$

• Apply $\exists$-rule to $\exists eats. (\neg plant \sqcap \neg dairy)$:
  Create new node $y$ and new edge $(x, y)$:
  $L(y) = \{\neg plant \sqcap \neg dairy\}; \quad L((x, y)) = eats$

• Apply $\forall$-rule to $\forall eats. plant$ in $L(x)$ and $L((x, y)) = eats$:
  Add $plant$ to $L(y)$
Example 1

- Apply $\sqcap$-rule to $\neg plant \sqcap \neg dairy$ in $L(y)$:
  Add $\{\neg plant, \neg dairy\}$ to $L(y)$: Clash
Example 1

- Apply $\cap$-rule to $\neg plant \cap \neg dairy$ in $L(y)$:
  Add $\{\neg plant, \neg dairy\}$ to $L(y)$: Clash

- Conclusion
  - Both applications of the $\sqcup$-rule lead to clashes
  - So vegan $\sqcap \neg vegetarian$ is unsatisfiable
  - So vegan $\sqsubseteq vegetarian$
Example 2

- Query: vegetarian $\sqsubseteq$ vegan
- Convert to: vegetarian $\sqcap \neg$vegan is satisfiable?
- Unfold and normalise vegetarian $\sqcap \neg$vegan:
  person $\sqcap \forall$eats.(plant $\sqcup$ dairy) $\sqcap$ ($\neg$person $\sqcup \exists$eats.$\neg$plant)
- Initialise $T$ to $L(x)$ to contain:
  \{person $\sqcap \forall$eats.(plant $\sqcup$ dairy) $\sqcap$ ($\neg$person $\sqcup \exists$eats.$\neg$plant)\}
Example 2

- Apply $\cap$-rule and add to $L(x)$:
  \[
  \{ \text{person}, \forall \text{eats.}\, (\text{plant} \sqcup \text{dairy}), \neg \text{person} \sqcup \exists \text{eats.}\, \neg \text{plant} \}
  \]
Example 2

- Apply $\cap$-rule and add to $L(x)$:
  \[
  \{ \text{person, } \forall \text{eats.}(\text{plant } \sqcup \text{dairy}), \neg \text{person } \sqcup \exists \text{eats.}\neg \text{plant} \} 
  \]

- Apply $\cup$-rule to $\neg \text{person } \sqcup \exists \text{eats.}\neg \text{plant}$:
  Add $\neg \text{person}$ to $L(x)$: Clash
  Go back and add $\exists \text{eats.}\neg \text{plant}$ to $L(x)$
Example 2

- Apply $\Box$-rule and add to $L(x)$:
  \[
  \{ \text{person}, \forall \text{eats.}(\text{plant} \sqcup \text{dairy}), \neg \text{person} \sqcup \exists \text{eats.} \neg \text{plant} \} 
  \]

- Apply $\sqcup$-rule to $\neg \text{person} \sqcup \exists \text{eats.} \neg \text{plant}$:
  Add $\neg \text{person}$ to $L(x)$: Clash
  Go back and add $\exists \text{eats.} \neg \text{plant}$ to $L(x)$

- Apply $\exists$-rule to $\exists \text{eats.} \neg \text{plant}$:
  Create new node $y$ and new edge $(x, y)$
  \[
  L(y) = \{ \neg \text{plant} \}; \ L((x, y)) = \text{eats}
  \]
Example 2

- Apply $\forall$-rule to $\forall eats.(plant \sqcap dairy)$ in $L(x)$ and $L((x, y)) = eats$:
  Add $plant \sqcap dairy$ to $L(y)$

- Apply $\sqcup$-rule to $plant \sqcap dairy$ in $L(y)$:
  Add $plant$ to $L(y)$: Clash
  Go back and add $dairy$ to $L(y)$

- Conclusion
  - No rules are applicable, so $T$ is fully expanded
  - So $\text{vegetarian} \sqcap \neg \text{vegan}$ is satisfiable
  - So $\text{vegetarian} \not\sqsubseteq \text{vegan}$
Example 2

- Apply $\forall$-rule to $\forall eats.(plant \sqcup dairy)$ in $L(x)$ and $L((x, y)) = eats$:
  
  Add $plant \sqcup dairy$ to $L(y)$

- Apply $\sqcup$-rule to $plant \sqcup dairy$ in $L(y)$:
  
  Add $plant$ to $L(y)$: Clash
  
  Go back and add $dairy$ to $L(y)$

- Conclusion

  - No rules are applicable, so $T$ is fully expanded

  - So vegetarian $\sqcup \neg$ vegan is satisfiable

  - So vegetarian $\not\sqsubseteq$ vegan
Example 2

- Apply \( \forall \)-rule to \( \forall \text{eats.}(\text{plant} \sqcup \text{dairy}) \) in \( L(x) \) and \( L((x, y)) = \text{eats} \):
  Add \( \text{plant} \sqcup \text{dairy} \) to \( L(y) \)

- Apply \( \sqcup \)-rule to \( \text{plant} \sqcup \text{dairy} \) in \( L(y) \):
  Add \( \text{plant} \) to \( L(y) \): Clash
  Go back and add \( \text{dairy} \) to \( L(y) \)

- Conclusion
  - No rules are applicable, so \( T \) is fully expanded
  - So \( \text{vegetarian} \sqcap \neg \text{vegan} \) is satisfiable
  - So \( \text{vegetarian} \nsubseteq \text{vegan} \)
The Brachman&Levesque DL and $\textit{ALC}$

<table>
<thead>
<tr>
<th>Constructor</th>
<th>B&amp;L</th>
<th>$\textit{ALC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conj.</td>
<td>$(\text{AND } A B)$</td>
<td>$A \sqcap B$</td>
</tr>
<tr>
<td>Univ. quant.</td>
<td>$(\text{ALL } R C)$</td>
<td>$\forall R.C$</td>
</tr>
<tr>
<td>Exist. quant.</td>
<td></td>
<td>$\exists R.C$</td>
</tr>
<tr>
<td>Unqual. exist. quant.</td>
<td>$(\text{EXISTS } 1 R)$</td>
<td>$\exists R.\top$</td>
</tr>
<tr>
<td>Number restriction</td>
<td>$(\text{EXISTS } n R)$</td>
<td></td>
</tr>
<tr>
<td>Role filler</td>
<td>$(\text{FILLS } R a)$</td>
<td></td>
</tr>
<tr>
<td>Assertion</td>
<td>$a \rightarrow C$</td>
<td>$C(a)$</td>
</tr>
</tbody>
</table>

- $\mathcal{FL}^-$ consists of Conj., Univ. quant., and Unqual. exist. quant.
- The B&L DL is slightly more general than $\mathcal{FL}^-$. 
- $\textit{ALC}$ is $\mathcal{FL}^-$ plus $\top$, $\bot$, and general negation. 
- The extension to $\textit{ALC}$ for a role filler would use $\forall R.\{a\}$. 
References

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