A Basic Logic: Classical Propositional Logic

CMPT 411/721
Propositional Logic

• Classical propositional logic is the best known logic and one of the simplest.
  • Aside: There are lots of propositional logics (which is why “classical” is used above) but when we talk about “propositional logic” we’ll mean the standard approach that you (should) have seen in earlier courses.

• Basic unit: *atoms* or *stomic sentences* which can be either true or false.
  • Strings like *a*, *p*, *john_is_happy*.

• These can be combined into more complex formulas by connectives like \( \land \) (and), \( \neg \) (not), and others.

• E.g. if *a* stands for “Robin is a student” and *b* for “Chris is a student” then \( a \lor b \) stands for “Robin is a student or Chris is a student”.
Propositional Logic

Given some domain, formulas will be either true or false.

- Consider a world (semester at SFU) in which CMPT310 is offered but CMPT354 is not. Let
  - \( p \) mean “CMPT310 is offered”
  - \( q \) mean “CMPT354 is offered”.

Then

- \( p \land q \) is not true but \( p \land \neg q \) is true.
- As well, \( p \lor q \) is true, and \( p \lor \neg p \) is true no matter what.

Given a set of formulas, we want to answer queries about these formulas.

- E.g. if \( r \) means “CMPT 411 is offered”, then from \( \{ p \lor q, \neg q \} \) we should be able to derive \( p \lor r \).
- In fact, the above inference holds no matter what \( p \), \( q \) and \( r \) mean.
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Logical Systems and Languages

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• The *syntax* defines the sentences in the language
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  - Sentences are most often referred to as *formulas*.
- The *semantics* defines the “meaning” of sentences;
  - i.e., defines *truth* of a *sentence* in a *world* or *domain*.

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A logic has three main parts:

- The syntax defines the sentences in the language
  - Sentences are most often referred to as formulas.
- The semantics defines the “meaning” of sentences;
  - i.e., defines truth of a sentence in a world or domain.
- Inference procedures (or a proof theory) define a means of deriving formulas from other formulas.
Propositional Logic Overview: Syntax

- Propositional logic is a simple logic – illustrates the main ideas
- We begin with the *proposition symbols* or *atomic sentences* or *atoms*: \( P = \{p, q, \ldots \} \).
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  - forming more complex sentences with
    - the unary operator \( \neg \) *(negation)* and
    - binary operators \( \wedge \) *(conjunction)*, \( \lor \) *(disjunction)*, \( \supset \) *(implication)*, \( \equiv \) *(biconditional)*.
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    - the unary operator \( \neg \) (*negation*) and
    - binary operators \( \land \) (*conjunction*), \( \lor \) (*disjunction*), \( \supset \) (*implication*), \( \equiv \) (*biconditional*)
  - \( \neg, \land, \lor, \supset, \equiv \) (and parentheses) are the *logical symbols*, whose meaning is fixed.
- Elements of \( P \) are the *nonlogical symbols*, whose meaning depends on the domain.
Propositional Logic Overview: Semantics

- For a formula, such as
  \((p \land \neg(q \lor (\neg p \supset r))) \lor (\neg p \lor r))\)
  we may want to be able to tell if the formula is true or false.

  In general this is impossible, unless we know whether each of
  \(p, q, r\) are true or false.

- However, if we know the truth value of every member of \(P\)
  then we can determine the truth value of any formula.
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- Simple recursive process evaluates an arbitrary formula.
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*Q*: If $P$ contains $n$ elements, how many interpretations are there?
# Truth tables for connectives

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$\neg P$</th>
<th>$P \land Q$</th>
<th>$P \lor Q$</th>
<th>$P \supset Q$</th>
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More Formally: The Language of PC

- Logical symbols: ¬, ∧, ∨, ⊃, ≡. (Parentheses are used for grouping.)
- Nonlogical symbols: Atomic sentences, \( P = p, q, r, \ldots \)
  - A literal is an atom or its negation.
- Sentences are defined recursively as follows:
- A string \( \phi \) is a sentence iff
  - \( \phi \) is an atomic formulae
  - \( \phi \) is of the form \( \neg \psi, (\psi \land \gamma), (\psi \lor \gamma), (\psi \supset \gamma), (\psi \equiv \gamma) \), where \( \psi \) and \( \gamma \) are sentences.
- If there is no ambiguity, parentheses may be omitted.
Semantics

- An *interpretation* for \( P \) is a mapping \( I : P \rightarrow \{true, false\} \).
  - In KR, sometimes the term *world* is used for *interpretation*, since an interpretation describes a possible state of the world.
- Given an interpretation, truth and falsity for all sentences is defined as follows:
  - \( I \models p \) iff \( I(p) = true \).
  - \( I \models \neg \phi \) iff \( I \not\models \phi \).
  - \( I \models \phi \land \psi \) iff \( I \models \phi \) and \( I \models \psi \).
  - \( I \models \phi \lor \psi \) iff \( I \models \phi \) or \( I \models \psi \).
  - \( I \models \phi \supset \psi \) iff \( I \not\models \phi \) or \( I \models \psi \).
  - \( I \models \phi \equiv \psi \) iff \((I \models \phi \iff I \models \psi)\).

It is more convenient (and less tedious) to take (say) \( \neg \) and \( \supset \) as primitive, and define the other connectives in terms of them.
Semantics: Key Terms

• For a formula \( \phi \), if there is an interpretation \( \mathcal{I} \) that makes \( \phi \) true, then
  • \( \phi \) is satisfiable, and
  • \( \mathcal{I} \) is model of \( \phi \), or \( \mathcal{I} \) satisfies \( \phi \), written \( \mathcal{I} \models \phi \).

⚠️ Be careful: Some texts use the term model for interpretation. (E.g. Russell and Norvig).

• Some formulas are true or false based on their form alone, and independent of an interpretation.
  • E.g. \( p \supset (q \supset p) \).

• A formula \( \phi \) that is true in every interpretation is called valid or, in propositional logic, a tautology.

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  • A formula that is false in every interpretation is \textit{unsatisfiable}. 

Semantics: Entailment

- *Entailment* means that one thing *follows from* another:
- Knowledge base $KB$ *entails* sentence $\phi$ if and only if:
  - $\phi$ is true in all interpretations/worlds in which $KB$ is true
  - Or: if $KB$ is true then $\phi$ must be true.
Semantics: Interpretations

- Write $KB \models \phi$ for $KB$ entails $\phi$.
  - So: $KB \models \phi$ iff for every interpretation $I$, if $I \models KB$ then $I \models \phi$.
  - Or: If $M(\phi)$ is the set of all models of $\phi$, then $KB \models \phi$ iff $M(KB) \subseteq M(\phi)$
- Note $\models$ is overloaded, since we also write $I \models \phi$.
- E.g., “The Leafs won” entails “The Leafs won or the Canucks won”.
- Entailment is a relationship between sentences (i.e., syntax) that is based on semantics
Example

Consider:

- If it rains John takes an umbrella
- If John takes an umbrella he doesn’t get wet
- If it doesn’t rain then John doesn’t get wet.

Show: John doesn’t get wet.
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Encode:

\[
\{ r \supset u, \quad u \supset \neg w, \quad \neg r \supset \neg w \}
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Query

$$
\{ r \Implies u, \ u \Implies \neg w, \ \neg r \Implies \neg w \} \models \neg w
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Prove that: \( \{ r \supset u, u \supset \neg w, \neg r \supset \neg w \} \models \neg w \)
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- Since in both cases \( \mathcal{I} \models \neg w \), we conclude \( \mathcal{I} \models \neg w \).
- We have shown for arbitrary \( \mathcal{I} \) that if \( \mathcal{I} \models \text{LHS} \) then \( \mathcal{I} \models \neg w \).
Discussion

- The preceding proof worked fine for showing a desired entailment.
- However, it is not clear how it would be generalised to arbitrary sets of formulas and queries.
- What we want is a **systematic** means of generating entailments.
- This is the subject of the next section.

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• Entailment says what things are implicitly true in a KB.
  • Inference is a procedure for computing entailments.
  • Key point: Inference is purely syntactic

\[ \text{KB} \vdash \phi \text{ means that } \phi \text{ can be derived from } \text{KB} \text{ by that procedure.} \]

\textbf{Desiderata:}

• \textbf{Soundness}: An inference procedure is sound if whenever \( \text{KB} \vdash \phi \), we also have \( \text{KB} \models \phi \).

• \textbf{Completeness}: An inference procedure is complete if whenever \( \text{KB} \models \phi \), we also have \( \text{KB} \vdash \phi \).
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  - **Completeness**: An inference procedure is complete if whenever $KB \models \phi$, we also have $KB \vdash \phi$. 

Inference

- An *inference procedure* is a (syntactic) procedure for deriving some formulas from others.
- In propositional logic, we can use entailment to derive conclusions by enumerating models.
  - I.e. can use entailment to do *inference*.
  - In first order logic we generally can’t enumerate all models
    - there may be infinitely many models, even for a finite knowledge base, and models may have infinite domains.
Inference by enumeration

In the following we compute whether $M(KB) \subseteq M(\phi)$.

PC.Entails?(KB, $\phi$) returns Boolean
Inputs:

- KB: the knowledge base, a sentence in propositional logic
- $\phi$: the query, a sentence in propositional logic

symbols $\leftarrow$ a list of the proposition symbols in KB and $\phi$

return Check(KB, $\phi$, symbols, []
Inference by enumeration

Check(KB, $\phi$, symbols, interp) returns Boolean

Inputs:
- KB: the knowledge base; $\phi$: the query
- symbols: atoms not yet assigned a truth value
- interp: a partial interpretation

if Empty?(symbols) then
  if Is.Model?(KB, interp) then return Is.Model?($\phi$, interp)
  else return true
else do
  $P \leftarrow$ First(symbols);
  rest $\leftarrow$ Rest(symbols)
  return Check(KB, $\phi$, rest, interp $\cup \{(P,\text{true})\}$) and
  Check(KB, $\phi$, rest, interp $\cup \{(P,\text{false})\}$)
• The procedure gives a depth-first enumeration of all interpretations
  • Hence, sound and complete
• Algorithm is $O(2^n)$ for $n$ symbols; problem is co-NP-complete
• If KB is empty, then the procedure computes whether $\phi$ is a tautology
  • This corresponds to the usual technique of checking truth tables.
Inference II: Axiomatic Proofs

If you have taken a course in logic, you’ve probably seen something like the following.

**Axiom Schemata:**

1. $\phi \supset (\psi \supset \phi)$
2. $(\phi \supset (\psi \supset \gamma)) \supset ((\phi \supset \psi) \supset (\phi \supset \gamma))$
3. $(\neg \psi \supset \neg \phi) \supset ((\neg \psi \supset \phi) \supset \psi)$

**Rule of Inference:**

Modus ponens: From $\phi$ and $\phi \supset \psi$ infer $\psi$

- Then (informally) $KB \vdash \phi$ just if there is a proof of $\phi$ from instances of the axioms and $KB$, using modus ponens.
• One can show that
  \[ KB \models \phi \iff KB \vdash \phi \]
• \textbf{Q:} So why not use such a system for doing proofs in KR?
One can show that

\[ KB \models \phi \ \text{iff} \ KB \vdash \phi \]

Q: So why not use such a system for doing proofs in KR?
A: It is almost impossible to control or direct the inference “towards” proving \( \phi \).
Inference III: Resolution

• Satisfiability is connected to inference via the following:
  \[ KB \models \phi \text{ if and only if } KB \cup \{ \neg \phi \} \text{ is unsatisfiable} \]
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- Resolution is a rule of inference defined for formulas in \textit{Conjunctive Normal Form} (CNF)
  - A formula or set of formulas is in CNF if it is expressed as a conjunction of disjunctions of literals
  - E.g., \((p \lor \neg q) \land (q \lor \neg r \lor \neg s)\).
    - Write as: \((p \lor \neg q)\), \((q \lor \neg r \lor \neg s)\)
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    - Write as: \( (p \lor \neg q), (q \lor \neg r \lor \neg s) \)
- A **clause** is a disjunction of literals.
- Can always express a set of formulas as a set of clauses.
Resolution

• *Resolution* inference rule:
  • Let \( p \lor A \) and \( \neg p \lor B \) be clauses (where \( A \) and \( B \) are arbitrary disjunctions of literals)
  • Then these clauses entail \( A \lor B \).

Resolution is sound and complete for propositional logic
• That is, a set of clauses is unsatisfiable iff \( \square \) can be obtained from those clauses via resolution.
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- **Resolution** inference rule:
  - Let $p \lor A$ and $\neg p \lor B$ be clauses (where $A$ and $B$ are arbitrary disjunctions of literals)
  - Then these clauses entail $A \lor B$.

- $A$ or $B$ can be empty:
  - $p$ and $\neg p \lor B$ entail $B$
  - $p$ and $\neg p$ entail $\square$ (a contradiction)

- Resolution is sound and complete for propositional logic
  - That is, a set of clauses is unsatisfiable iff $\square$ can be obtained from those clauses via resolution.
Using Resolution to Compute Entailments

To show whether $KB \models \phi$, show instead that $KB \cup \{\neg \phi\}$ is unsatisfiable:

1. Convert $KB \cup \{\neg \phi\}$ into conjunctive normal form.
2. Use resolution to determine whether $KB \cup \{\neg \phi\}$ is unsatisfiable.
3. If so then $KB \models \phi$; otherwise $KB \not\models \phi$.

We’ll cover resolution in detail in going over first-order logic (next).
Summary

- Propositional logic is often seen as lacking in expressive power.
- As well, determining satisfiability and entailment are NP-complete and co-NP-complete respectively.
  - I.e. it seems unlikely that there will be efficient procedures for these tasks.
- Nonetheless, the concepts behind propositional logic are common to (pretty much) all other logics.
- As well, in the last ~20 years, very efficient satisfiability (SAT) solvers have been developed.
  - This has led to huge advances in areas such as verification and model checking.
Summary (continued)

- Also, for applications expressed in first-order logic over a finite domain, one can “compile” the FO theory into propositional logic, and run a SAT solver.
  - This is done via a procedure of “grounding” whereby variables and quantifiers are eliminated.
  - Areas such as planning have been addressed in this way.