CMPT 825
NLP

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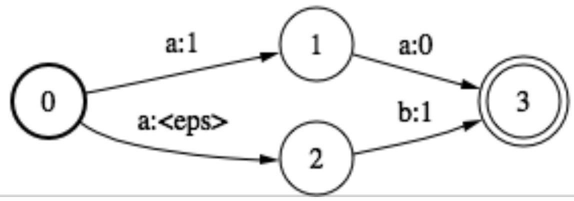
Finite-state transducers

- a : 0 is a notation for a mapping between two alphabets $a \in \Sigma_1$ and $0 \in \Sigma_2$
- Finite-state transducers (FSTs) accept pairs of strings
- Finite-state automata equate to regular languages and FSTs equate to regular relations
- e.g. $L = \{ (x^n, y^n) : n > 0, x \in \Sigma_1 \text{ and } y \in \Sigma_2 \}$ is a regular relation accepted by some FST. It maps a string of *x*'s into an equal length string of *y*'s

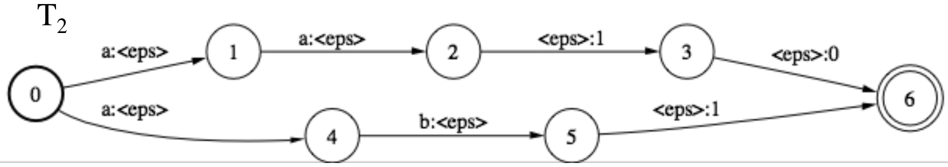
Finite-state transducers

T_1

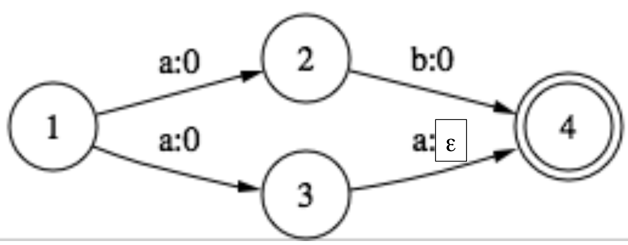


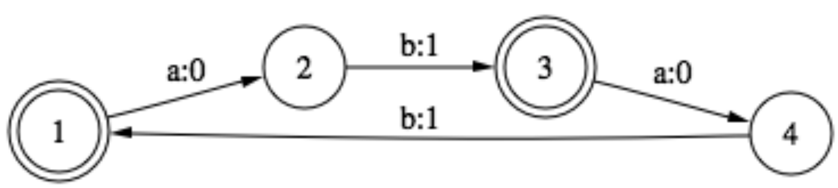
$R(T_1) = R(T_2) = \{ (aa, 10), (ab, 1) \}$

T_2

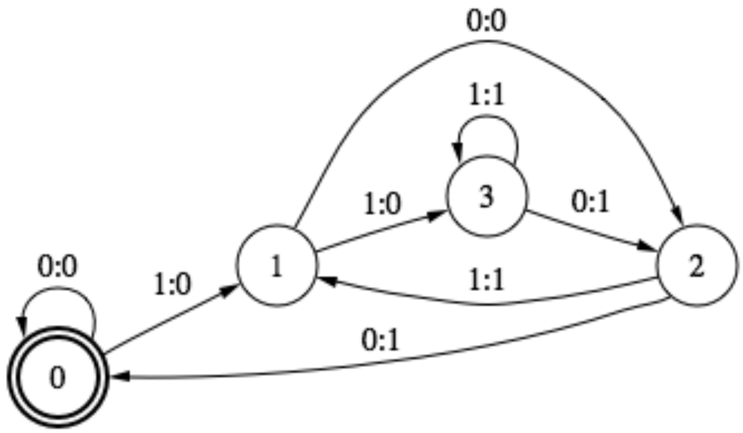


Finite-state transducers





Finite-state transducers



Regular relations

- A generalization of regular languages
- The set of regular relations is:
  - The empty set and (x,y) for all $x, y \in \Sigma_1 \times \Sigma_2$ is a regular relation
  - If R_1, R_2 and R are regular relations then:

    $$R_1 \cdot R_2 = \{(x_1x_2, y_1y_2) \mid (x_1, y_1) \in R_1, (x_2, y_2) \in R_2\}$$

    $$R_1 \cup R_2$$

    $$R^* = \cup_{i=0}^{\infty} R_i$$

  - There are no other regular relations

Finite-state transducers

- Formal definition:
  - Q: finite set of states, $q_0, q_1, \ldots, q_n$
  - Σ: alphabet composed of input/output pairs i:o where $i \in \Sigma_1$ and $o \in \Sigma_2$ and so $\Sigma \subseteq \Sigma_1 \times \Sigma_2$
  - q_0: start state
  - F: set of final states
  - $\delta(q, i{:}o)$ is the transition function which returns a set of states

Finite-state transducers: Examples

- (a^n, b^n): map *n a*'s into *n b*'s
- rot13 encryption (the Caesar cipher): assuming 26 letters each letter is mapped to the letter 13 steps ahead (mod 26), e.g. *cipher* → *pvcure*
- reversal of a fixed set of words
- reversal of all strings upto fixed length *k*
- input: binary number *n*, and output: binary number *n+1*
- upcase or lowercase a string of any length
- *Pig latin: *pig latin is goofy* → *igpay atinlay is oofygay*
- *convert numbers into pronunciations,

  e.g. 230.34 two hundred and thirty point three four

Finite-state transducers

- Following relations are cannot be expressed as a FST
  - ($a^n b^n$, c^n): because $a^n b^n$ is not regular
  - reversal of strings of any length
  - $a^i b^j \rightarrow b^j a^i$ for any *i, j*
- Unlike regular languages, regular relations are not closed under intersection
  - ($a^n b^*$, c^n) $\cap$ ($a^* b^n$, c^n) produces ($a^n b^n$, c^n)
  - However, regular relations with input and output of equal lengths **are** closed under intersection

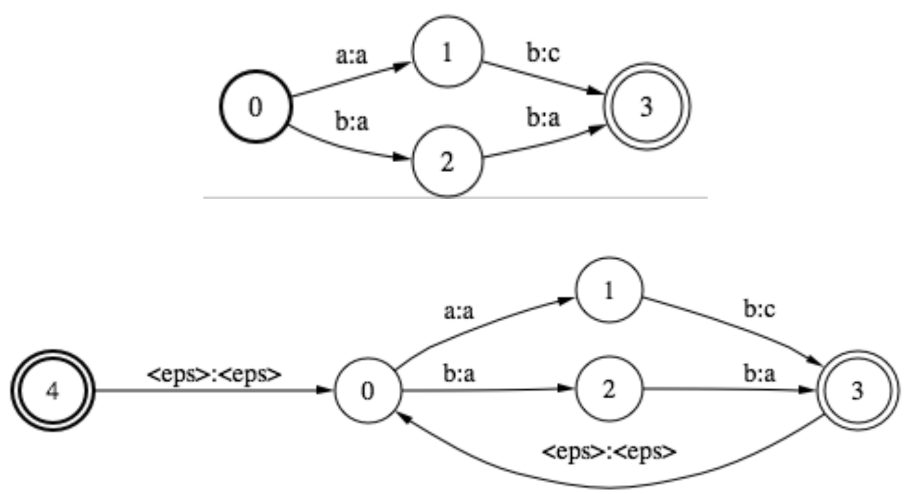
Regular Relations Closure Properties

- Regular relations (rr) are ***closed*** under some operations
- For example, if R_1, R_2 are regular relns:
  - union ($R_1 \cup R_2$ results in R_3 which is a rr)
  - concatenation
  - iteration (R_1+ = one or more repeats of R_1)
  - Kleene closure (R_1* = zero or more repeats of R_1)
- However, unlike regular languages, regular relns are not closed under:
  - intersection (possible for equal length regular relns)
  - complement

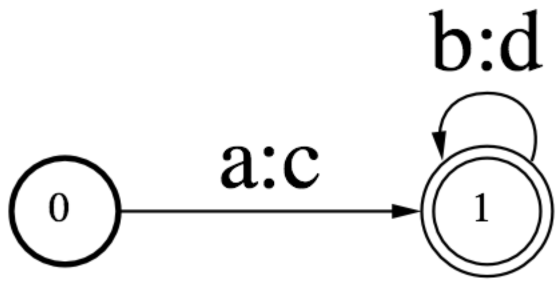
Regular Relations Closure Properties

- New operations for regular relations:
  - composition
  - project input (or output) language to regular language; for FST t, input language = $\pi_1(t)$, output = $\pi_2(t)$
  - take a regular language and create the identity regular relation; for FSM f, let FST for identity relation be Id(f)
  - take two regular languages and create the cross product relation; for FSMs f & g, FST for cross product is $f \times g$
  - take two regular languages, and mark each time the first language matches any string in the second

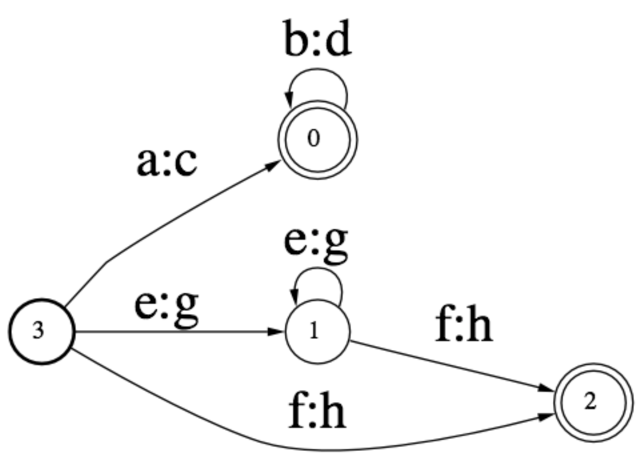
Regular Relation/FST Kleene Closure



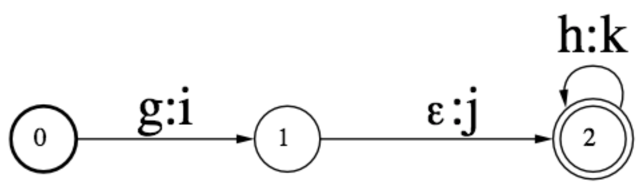
Regular Expressions for FSTs



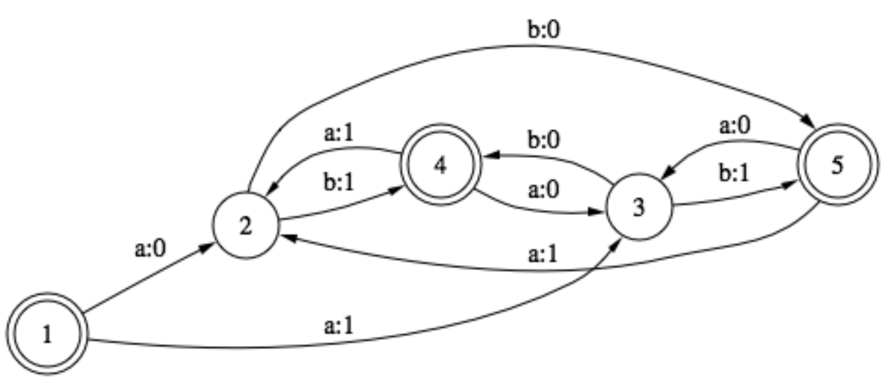
(a:c) (b:d)*



(a:c (b:d)*) | ((e:g)* f:h)



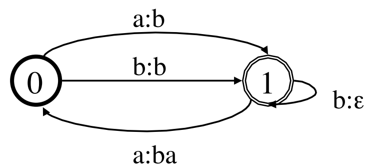
g:i ε:j (h:k)*



((a:0 | a:1) (b:0 | b:1))*

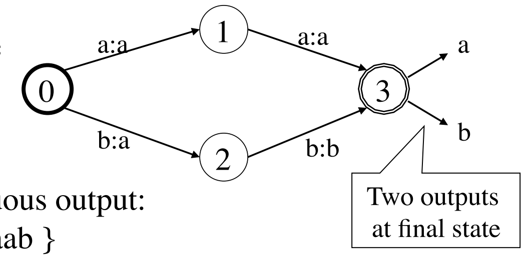
Subsequential FSTs

Sequential transducer = transducer with deterministic input



input: abbaa   output: bbab

p-subsequential transducer = transducer with at most *p* output strings at each final state

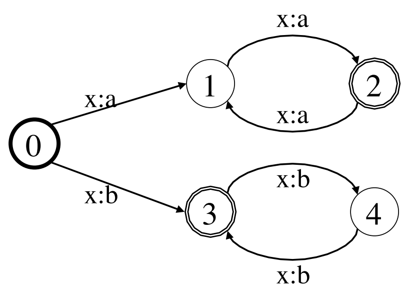


input: aa   ambiguous output: { aaa, aab }

Subsequential FSTs

- Consider an FST in which for every symbol scanned from the input we can deterministically choose a path and produce an output
- Such an FST is analogous to a deterministic FSM. It is called a **subsequential** FST.
- Subsequential transducers with *p* outputs on the final state is called a ***p*-subsequential** FST
- p-subsequential FSTs can produce ambiguous outputs for a given input string

FST that is not subsequential



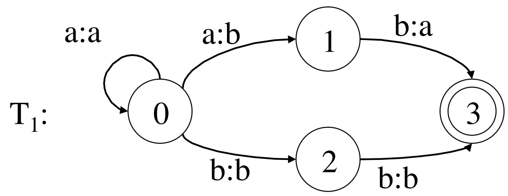
Input: x^n
Output: a^n if n is even, else b^n

FST Algorithms

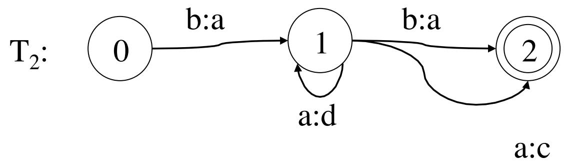
- **Compose**: Given two FSTs *f* and *g* defining regular relations R_1 and R_2 create the FST $f \circ g$ that computes the composition: $R_1 \circ R_2$
- **Recognition**: Is a given pair of strings accepted by FST *t*?
- **Transduce**: given an input string, provide the output string(s) as defined by the regular relation provided by an FST

Composing FSTs

on input side:
a^n == a*



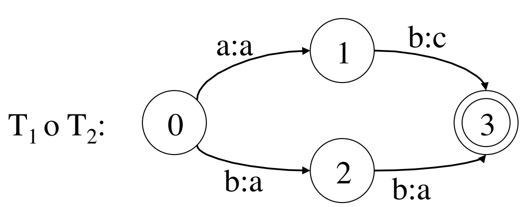
a^n ab := a^n ba
a^n bb := a^n bb



b a^n b := a d^n a
b a^n a := a d^n c

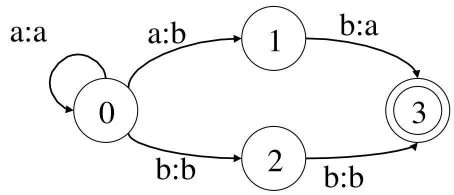
What is T_1 composed with T_2, aka T_1 o T_2 ?

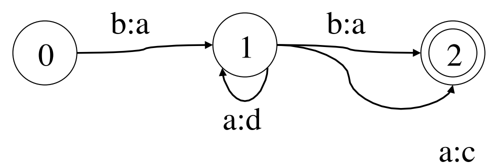
Composing FSTs



ab := ac
bb := aa

Composing FSTs



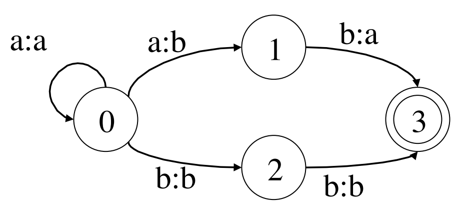


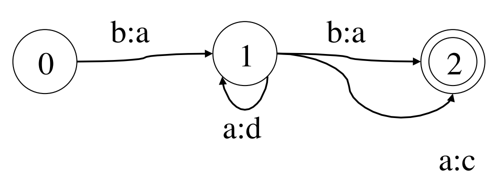
0 1 a : b	0 1 b : a
0 2 b : b	1 2 b : a
2 3 b : b	

0 0 a : a	1 1 a : d
1 3 b : a	1 2 a : c

(0,0) (1,1) a : a   (0,0) (2,1) b : a   (0,1) (0,1) a : d   (1,1) (3,1) b : d
(0,1) (1,2) a : a   (0,1) (2,2) b : a   (0,1) (0,2) a : c   (1,1) (3,2) b : c
(2,0) (3,1) b : a   (2,1) (3,2) b : a

Composing FSTs





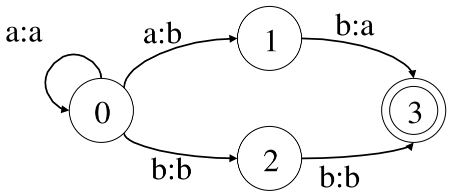
0 1 a : b	0 1 b : a
0 2 b : b	1 2 b : a
2 3 b : b	

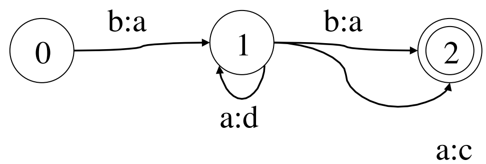
0 0 a : a	1 1 a : d
1 3 b : a	1 2 a : c

(0,0) (1,1) a : a   (0,0) (2,1) b : a   (0,1) (0,1) a : d   (1,1) (3,1) b : d
(0,1) (1,2) a : a   (0,1) (2,2) b : a   (0,1) (0,2) a : c   <u>(1,1) (3,2) b : c</u>
(2,0) (3,1) b : a   <u>(2,1) (3,2) b : a</u>

start with pair of final states

Composing FSTs

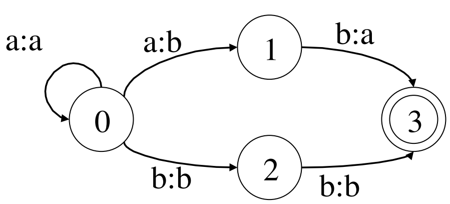


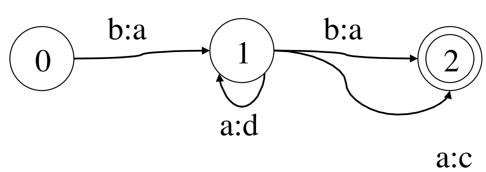


0 1 a : b 0 2 b : b 2 3 b : b	0 1 b : a 1 2 b : a	0 0 a : a 1 3 b : a	1 1 a : d 1 2 a : c

(0,0) (1,1) a : a   (0,0) (2,1) b : a   (0,1) (0,1) a : d   (1,1) (3,1) b : d
(0,1) (1,2) a : a   (0,1) (2,2) b : a   (0,1) (0,2) a : c   (1,1) (3,2) b : c
(2,0) (3,1) b : a   (2,1) (3,2) b : a

Composing FSTs



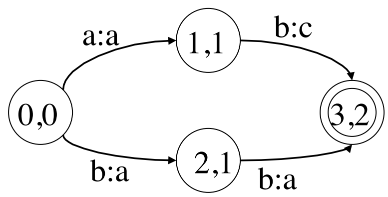


0 1 a : b 0 2 b : b 2 3 b : b	0 1 b : a 1 2 b : a	0 0 a : a 1 3 b : a	1 1 a : d 1 2 a : c

(0,0) (1,1) a : a   (0,0) (2,1) b : a   (0,1) (0,1) a : d   (1,1) (3,1) b : d
(0,1) (1,2) a : a   (0,1) (2,2) b : a   (0,1) (0,2) a : c   (1,1) (3,2) b : c
(2,0) (3,1) b : a   (2,1) (3,2) b : a

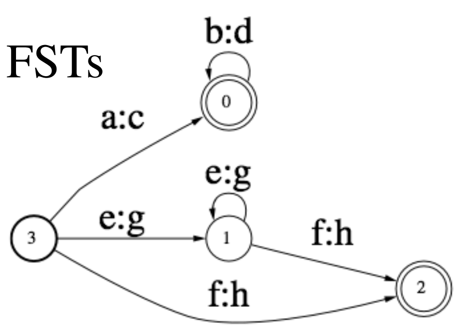
Composing FSTs

$T_1 \circ T_2$:



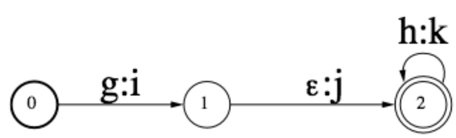
ab := ac
bb := aa

Composing FSTs

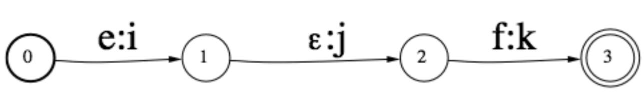


(a:c (b:d)*) | ((e:g)* f:h)

g:i ε:j (h:k)*



e:i ε:j f:k



FST Composition

- Input: transducer S and T
- Transducer composition results in a new transducer with states and transitions defined by matching compatible input-output pairs:

  match(s,t) =

  $\{ (s,t) \rightarrow^{x:z} (s',t') : s \rightarrow^{x:y} s' \in S.edges \text{ and } t \rightarrow^{y:z} t' \in T.edges \} \cup$

  $\{ (s,t) \rightarrow^{x:\varepsilon} (s',t) : s \rightarrow^{x:\varepsilon} s' \in S.edges \} \cup$

  $\{ (s,t) \rightarrow^{\varepsilon:z} (s,t') : t \rightarrow^{\varepsilon:z} t' \in T.edges \}$
- Correctness: any path in composed transducer mapping *u* to *w* arises from a path mapping *u* to *v* in S and path mapping *v* to *w* in T, for some *v*

Complex FSTs with composition

- Take, for example, the task of constructing an FST for the Soundex algorithm
- Soundex is useful to map spelling variants of proper names to a single code (hashing names)
- It depends on a mapping from letters to codes

Soundex

- Mapping from letters to numbers:

  b, f, p, v → 1

  c, g, j, k, q, s, x, z → 2

  d, t → 3

  l → 4

  m, n → 5

  r → 6

Soundex

- The Soundex algorithm:
  - If two or more letters with the same number are adjacent in the input, or adjacent with intervening *h*'s or *w*'s omit all but the first
  - Retain the first letter and delete all occurrences of *a, e, h, i, o, u, w, y*
  - Except for the first letter, change all letters into numbers
  - Convert result into LNNN (letter and 3 numbers), either truncate or add 0s

Soundex

- Example:

  Losh-shkan, Los-qam

  Loshhkan, Losqam

  Lskn, Lsqm

  L225, L225

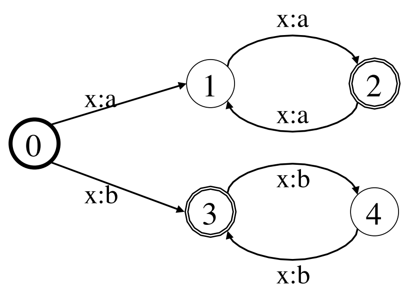
- Other examples:

  Euler (E460), Gauss (G200), Hilbert (H416), **Knuth** (K530), Lloyd (L300), Lukasiewicz (L222), and Wachs (W200)

Soundex

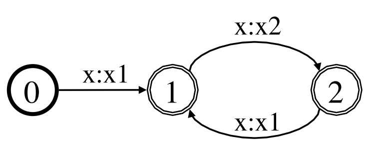
- How can we implement Soundex as a FST?
- For each step in Soundex, the FST is quite simple to write
- Writing a single FST from scratch that implements Soundex is quite challenging
- A simpler solution is to build small FSTs, one for each step, and then use FST composition to build the FST for Soundex

FST that is not subsequential

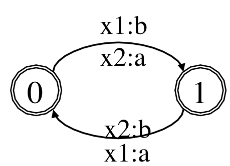


Input: x^n
Output: a^n if n is even, else b^n

Conversion to subsequential FST



o FSM-Reverse o



Input: x^n
- Step1 output: *(x1|x2)*x2* if n is even, else *(x1|x2)*x1*
- Step2 output: reversal of Step1 output
- Step3 output: a^n if n is even, else b^n

Interesting fact: this can be done for any non-subsequential FST to convert it into a subsequential FST

Recognition of string pairs

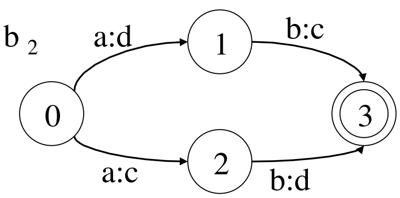
```
function FSTRecognize (input[], output[], q):
    Agenda = { (start-state, 0, 0) }
    Current = (state, i, o) = pop(Agenda) // i :- inputIndex, o :- outputIndex
    while (true) {
        if (Current is an accept item) return accept
        else Agenda = Agenda ∪ GenStates(q, state, input, output, i, o)
        if (Agenda is empty) return reject
        else Current = (state, i, o) = pop(Agenda)
    }
function GenStates (q, state, input[], output[], i, o):
    return { (q', i, o) : for all q' = q(state, ε:ε) } ∪
           { (q', i, o+1) : for all q' = q(state, ε:output[o+1]) } ∪
           { (q', i+1, o) : for all q' = q(state, input[i+1]:ε) } ∪
           { (q', i+1, o+1) : for all q' = q(state, input[i+1], output[i+1]) }
```


Transduction: input → output

- The **transduce** operation for a FST *t* can be simulated efficiently using the following steps:
  1. Convert the input string into a FSM *f* (the machine only accepts the input string, nothing else).
  2. Convert *f* into a FST by taking Id(*f*) and compose with *t* to give a new FST *g* = Id(*f*) o *t*. (note that *g* only contains those paths compatible with input *f*)
  3. Finally project the output language of *g* to give a FSM for the output of transduce: $\pi_2(g)$
  4. Optionally, eliminate any transitions that only derive the empty string from the $\pi_2(g)$ FST.
- What follows is an alternate version that attempts to produce all output strings

Transduction: input → output

input: $_0$ a $_1$ b $_2$



agenda: { (0, 0, []) }

agenda: { (1, 1, [d]), (2, 1, [c]) }

agenda: { (2, 1, [c]), (3, 2, [d ⊕ c]) }

agenda: { (3, 2, [d ⊕ c, c ⊕ d]) }

agenda: { (3, 2, [dc, cd]) }

(3, 2, [dc, cd]) is an *accept* item: output = dc, cd

Transduction: input → output

```
function FSTtransduce (input[], q):
    Agenda = { (start-state, 0, []) } // each item contains list of partial outputs
    Current = (state, i, out) = pop(Agenda) // i :- inputIndex, out :- output-list
    output = ()
    while (true) {
        if (Current is an accept item) output ⊕ out
        else Agenda = Agenda ∪ GenStates(q, state, input, out, i)
        if (Agenda is empty) return output
        else Current = (state, i, o) = pop(Agenda)
    }
```

Transduction: input → output

```
function FSTtransduce (input[], q):
    Agenda = { (start-state, 0, []) } // each item contains list of partial outputs
    Current = (state, i, out) = pop(Agenda) // i :- inputIndex, out :- output-list
    output = ()
    while (true) {
        if (Current is an accept item) output ⊕ out
        else Agenda = Agenda ∪ GenStates(q, state, input, out, i)
        if (Agenda is empty) return output
        else Current = (state, i, o) = pop(Agenda)
    }
```

∪ adds new output to output lists in items seen before

Transduction: input → output

```
function FSTtransduce (input[], q):
    Agenda = { (start-state, 0, []) } // each item contains list of partial outputs
    Current = (state, i, out) = pop(Agenda) // i :- inputIndex, out :- output-list
    output = ()
    while (true) {
        if (Current is an accept item) output ⊕ out
        else Agenda = Agenda ∪ GenStates(q, state, input, out, i)
        if (Agenda is empty) return output
        else Current = (state, i, o) = pop(Agenda)
    }
function GenStates (q, state, input[], out, i):
    return { (q', i, out) : for all q' = q(state, ε:ε) } ∪
        { (q', i, out ⊕ newOut) : for all q' = q(state, ε:newOut) } ∪
        { (q', i+1, out) : for all q' = q(state, input[i+1]:ε) } ∪
        { (q', i+1, out ⊕ newOut) : for all q' = q(state, input[i+1], newOut) }
```

Transduction: input → output

```
function FSTtransduce (input[], q):
    Agenda = { (start-state, 0, []) } // each item contains list of partial outputs
    Current = (state, i, out) = pop(Agenda) // i :- inputIndex, out :- output-list
    output = ()
    while (true) {
        if (Current is an accept item) output ⊕ out
        else Agenda = Agenda ∪ GenStates(q, state, input, out, i)
        if (Agenda is empty) return output
        else Current = (state, i, o) = pop(Agenda)
    }
function GenStates (q, state, input[], out, i):
    return { (q', i, out) : for all q' = q(state, ε:ε) } ∪
        { (q', i, out ⊕ newOut) : for all q' = q(state, ε:newOut) } ∪
        { (q', i+1, out) : for all q' = q(state, input[i+1]:ε) } ∪
        { (q', i+1, out ⊕ newOut) : for all q' = q(state, input[i+1], newOut) }
```

⊕ concatenates new output string to each item in out (the output list for each item)

Cross-product FST

- For regular languages L_1 and L_2, we have two FSAs, M_1 and M_2

$$M_1 = (\Sigma, Q_1, q_1, F_1, \delta_1)$$
$$M_2 = (\Sigma, Q_2, q_2, F_2, \delta_2)$$

- Then a transducer accepting $L_1 \times L_2$ is defined as:

$$T = (\Sigma, Q_1 \times Q_2, \langle q_1, q_2 \rangle, F_1 \times F_2, \delta)$$
$$\delta(\langle s_1, s_2 \rangle, a, b) = \delta_1(s_1, a) \times \delta_2(s_2, b)$$
$$\text{for any } s_1 \in Q_1, s_2 \in Q_2 \text{ and } a, b \in \Sigma \cup \{\varepsilon\}$$

Summary

- Finite state transducers specify regular relations
  - Encoding problems as finite-state transducers
- Extension of regular expressions to the case of regular relations/FSTs
- FST closure properties: union, concatenation, composition
- FST special operations:
  - creating regular relations from regular languages (Id, cross-product);
  - creating regular languages from regular relations (projection)
- FST algorithms
  - Recognition, Transduction
  - Determinization, Minimization? (not all FSTs can be determinized)