Integers I

http://xkcd.com/257/

Acknowledgments: These slides have been modified by Arrvindh Shriraman, Justin Tsia

Roadmap

C:

```c
car *c = malloc(sizeof(car));
c->miles = 100;
c->gals = 17;
float mpg = get_mpg(c);
free(c);
```

Java:

```java
Car c = new Car();
c.setMiles(100);
c.setGals(17);
float mpg =
    c.getMPG();
```

Memory & data
Arrays & structs
Integers & floats
RISC V assembly
Procedures & stacks
Executables
Memory & caches
Processor Pipeline
Performance
Parallelism

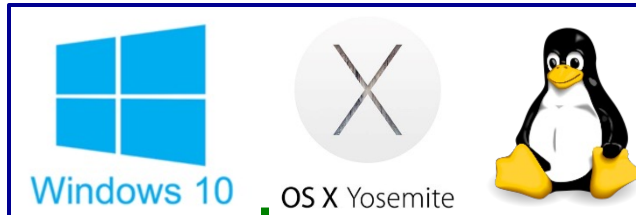
Assembly language:

```
get_mpg(car*):
        lw      a5,0(a0)
        lw      a4,4(a0)
        divw    a5,a5,a4
        fcvt.s.w        fa0,a5
        ret
```

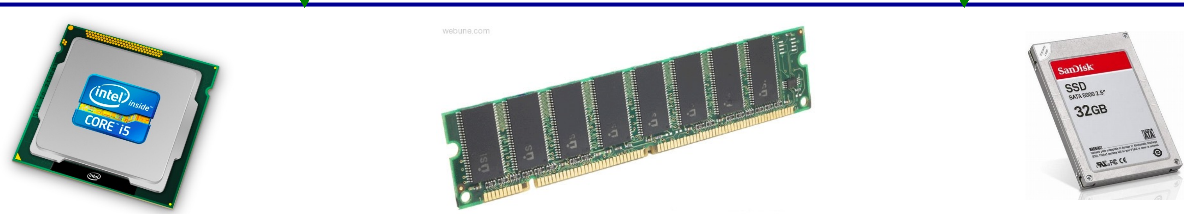
Machine code:

```
0111010000011000
100011010000010000000010
1000100111000010
110000011111101000011111
```

OS:

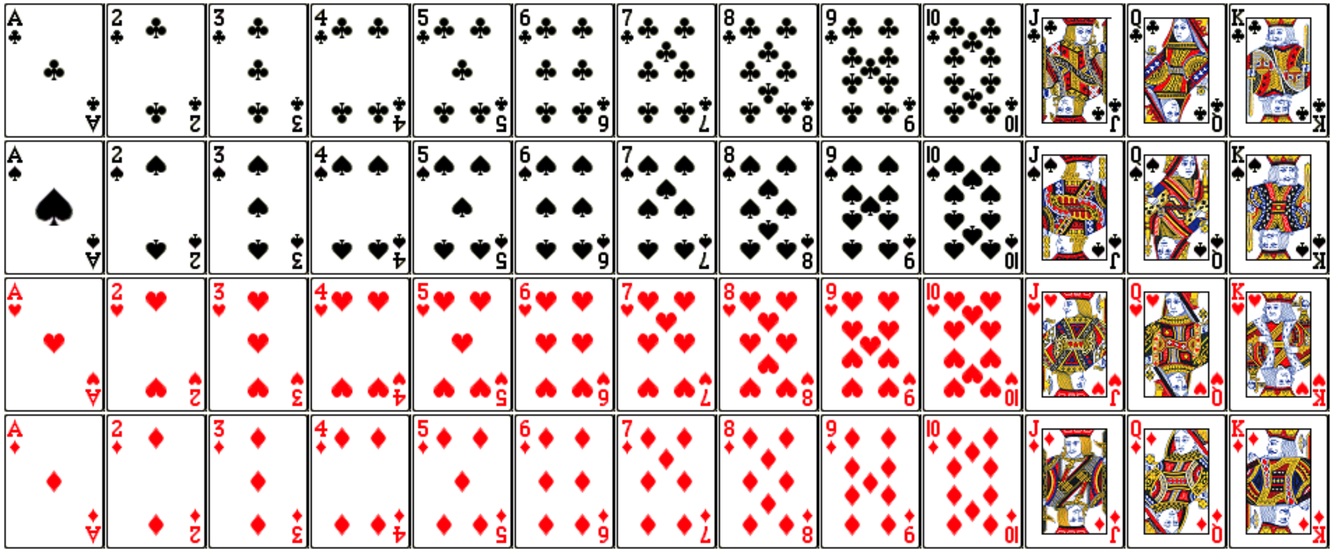


Computer system:



But before we get to integers….

- Encode a standard deck of playing cards
- 52 cards in 4 suits
  - How do we encode suits, face cards?
- What operations do we want to make easy to implement?
  - Which is the higher value card?
  - Are they the same suit?



Boolean Algebra

- Developed by George Boole in 19th Century
  - Algebraic representation of logic (True → 1, False → 0)
  - AND: `A&B=1` when both A is 1 and B is 1
  - OR: `A|B=1` when either A is 1 or B is 1
  - XOR: `A^B=1` when either A is 1 or B is 1, but not both
  - NOT: `~A=1` when A is 0 and vice-versa
  - DeMorgan's Law: `~(A|B) = ~A & ~B`
    `~(A&B) = ~A | ~B`

AND

&	0	1
0	**0**	**0**
1	**0**	**1**

OR

|	0	1
0	**0**	**1**
1	**1**	**1**

XOR

^	0	1
0	**0**	**1**
1	**1**	**0**

NOT

~	
0	**1**
1	**0**

General Boolean Algebras

- Operate on bit vectors
  - Operations applied bitwise
  - All of the properties of Boolean algebra apply

```
    01101001       01101001       01101001
  & 01010101     | 01010101     ^ 01010101     ~ 01010101
```

- Examples of useful operations:

$$x \,\hat{}\, x = 0$$

```
  01010101
^ 01010101
  00000000
```

$$x \mid 1 = 1, \quad x \mid 0 = x$$

```
  01010101
| 11110000
  11110101
```

Bit-Level Operations in C

- `&` (AND), `|` (OR), `^` (XOR), `~` (NOT)
  - View arguments as bit vectors, apply operations bitwise
  - Apply to any "integral" data type
    - `long`, `int`, `short`, `char`, `unsigned`
- Examples with `char a, b, c;`
  - ```
    a = (char) 0x41;    // 0x41->0b 0100 0001
    b = ~a;             //       0b           ->0x
    ```
  - ```
    a = (char) 0x69;    // 0x69->0b 0110 1001
    b = (char) 0x55;    // 0x55->0b 0101 0101
    c = a & b;          //       0b           ->0x
    ```
  - ```
    a = (char) 0x41;    // 0x41->0b 0100 0001
    b = a;              //       0b 0100 0001
    c = a ^ b;          //       0b           ->0x
    ```
    ```

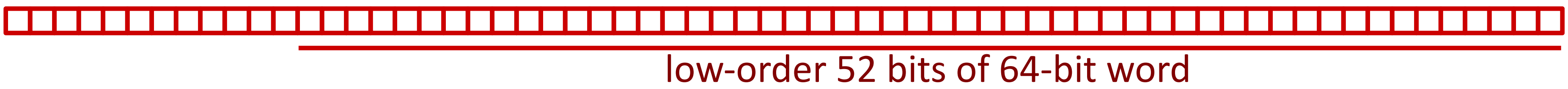
Contrast: Logic Operations

- Logical operators in C: `&&` (AND), `||` (OR), `!` (NOT)
  - **<u>0</u>** is False, **<u>anything nonzero</u>** is True
  - **<u>Always</u>** return 0 or 1
  - Early termination (a.k.a. short-circuit evaluation) of `&&`, `||`
    - `int x = (42 == 6) || boom();`
    - `int y = (42 == 6) && boom();`

- Examples (`char` data type)
  - `!0x41  ->  0x00`
  - `!0x00  ->  0x01`
  - `!!0x41 ->  0x01`
  - `0xCC && 0x33 -> 0x01`
  - `0x00 || 0x33 -> 0x01`

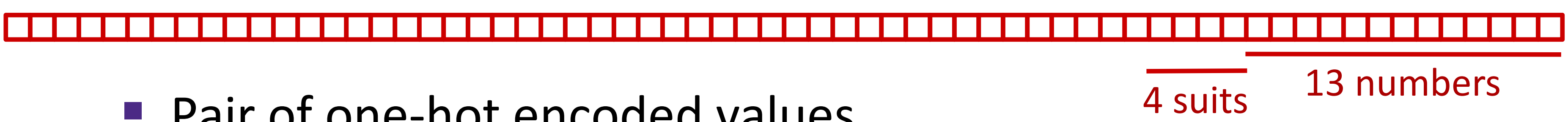
Two possible representations

1) 1 bit per card (52): bit corresponding to card set to 1



- "One-hot" encoding (similar to set notation)
- Drawbacks:
    - Hard to compare values and suits
    - Large number of bits required

2) 1 bit per suit (4), 1 bit per number (13): 2 bits set

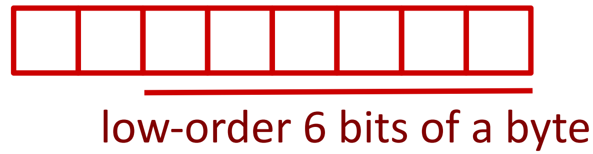


- Pair of one-hot encoded values
- Easier to compare suits and values, but still lots of bits used

Two better representations

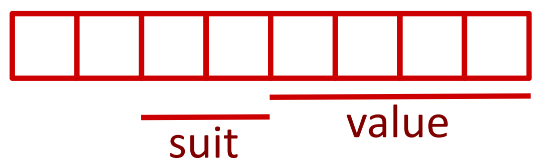
3) Binary encoding of all 52 cards – only 6 bits needed

- $2^6 = 64 \geq 52$



- Fits in one byte (smaller than one-hot encodings)
- How can we make value and suit comparisons easier?

4) Separate binary encodings of suit (2 bits) and value (4 bits)



♣	00
♦	01
♥	10
♠	11

- Also fits in one byte, and easy to do comparisons

K	Q	J	. . .	3	2	A
1101	1100	1011	. . .	0011	0010	0001

Compare Card **Suits**

mask: a bit vector designed to achieve a desired behavior when used with a bitwise operator on another bit vector *v*.
Here we turns all *but* the bits of interest in *v* to 0.

```
char hand[5];         // represents a 5-card hand
char card1, card2;    // two cards to compare
card1 = hand[0];
card2 = hand[1];
...
if ( isSameSuit(card1, card2) ) { ... }
```

```
#define SUIT_MASK  0x30

int isSameSuit(char card1, char card2) {
  return (card1 & SUIT_MASK) == (card2 & SUIT_MASK);
}
```

returns `int`

SUIT_MASK = 0x30 =

0	0	1	1	0	0	0	0

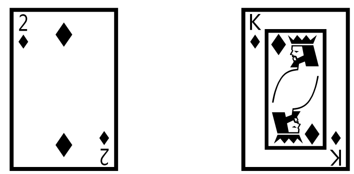
suit value

Compare Card **Suits**

mask: a bit vector designed to achieve a desired behavior when used with a bitwise operator on another bit vector *v*.
Here we turns all *but* the bits of interest in *v* to 0.

```
#define SUIT_MASK  0x30

int isSameSuit(char card1, char card2) {
  return (!((card1 & SUIT_MASK) ^ (card2 & SUIT_MASK)));
  // return (card1 & SUIT_MASK) == (card2 & SUIT_MASK);
}
```



0	0	0	1	0	0	1	0

&

0	0	1	1	0	0	0	0

=

0	0	0	1	0	0	0	0

SUIT_MASK

0	0	0	1	1	1	0	1

&

0	0	1	1	0	0	0	0

=

0	0	0	1	0	0	0	0

^

0	0	0	0	0	0	0	0

!

0	0	0	0	0	0	0	1

`!(x^y)` equivalent to `x==y`

Compare Card Values

mask: a bit vector designed to achieve a desired behavior when used with a bitwise operator on another bit vector *v*.

```
char hand[5];         // represents a 5-card hand
char card1, card2;    // two cards to compare
card1 = hand[0];
card2 = hand[1];
...
if ( greaterValue(card1, card2) ) { ... }
```

```
#define VALUE_MASK  0x0F

int greaterValue(char card1, char card2) {
  return ((unsigned char)(card1 & VALUE_MASK) >
          (unsigned char)(card2 & VALUE_MASK));
}
```

VALUE_MASK = 0x0F =

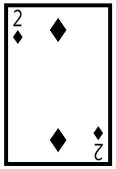
0	0	0	0	1	1	1	1
		suit		value			

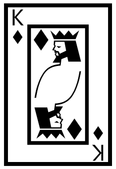
Compare Card **Values**

mask: a bit vector designed to achieve a desired behavior when used with a bitwise operator on another bit vector *v*.

```
#define VALUE_MASK  0x0F

int greaterValue(char card1, char card2) {
  return ((unsigned int)(card1 & VALUE_MASK) >
          (unsigned int)(card2 & VALUE_MASK));
}
```





0	0	1	0	0	0	1	0

&

0	0	0	0	1	1	1	1

=

0	0	0	0	0	0	1	0

VALUE_MASK

0	0	1	0	1	1	0	1

&

0	0	0	0	1	1	1	1

=

0	0	0	0	1	1	0	1

$2_{10} > 13_{10}$

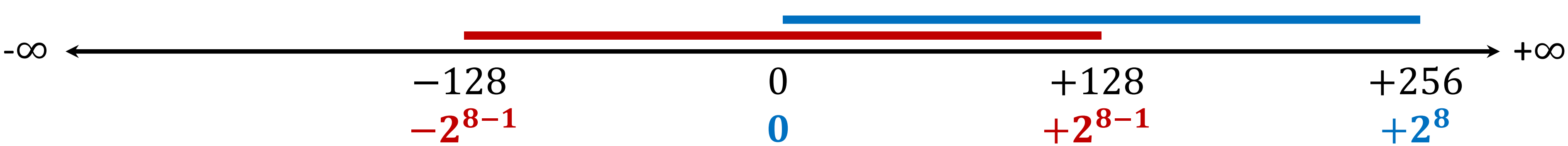
0 (false)

Integers

- **Binary representation of integers**
  - **Unsigned and signed**
  - Casting in C
- Consequences of finite width representation
  - Overflow, sign extension
- Shifting and arithmetic operations

Encoding Integers

- The hardware (and C) supports two flavors of integers
  - *unsigned* – only the non-negatives
  - *signed* – both negatives and non-negatives
- Cannot represent all integers with w bits
  - Only 2^w distinct bit patterns
  - Unsigned values: $0 \ldots 2^w-1$
  - Signed values: $-2^{(w-1)} \ldots 0 \ldots 2^{(w-1)}-1$
- **Example:** 8-bit integers (*e.g.* `char`)



Unsigned Integers

- Unsigned values follow the standard base 2 system
  - $b_7b_6b_5b_4b_3b_2b_1b_0 = b_7 2^7 + b_6 2^6 + \cdots + b_1 2^1 + b_0 2^0$
- Add and subtract using the normal "carry" and "borrow" rules, just in binary

```
  63
+  8
  71
```

```
  00111111
+ 00001000
  01000111
```

- Useful formula: N ones in a row = $2^N - 1$
- How would you make *signed* integers?

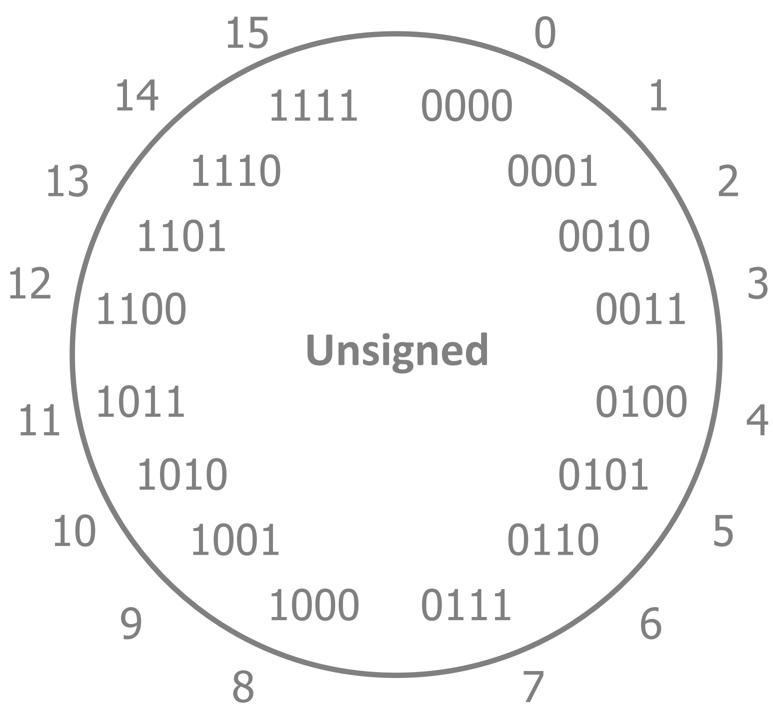
Sign and Magnitude

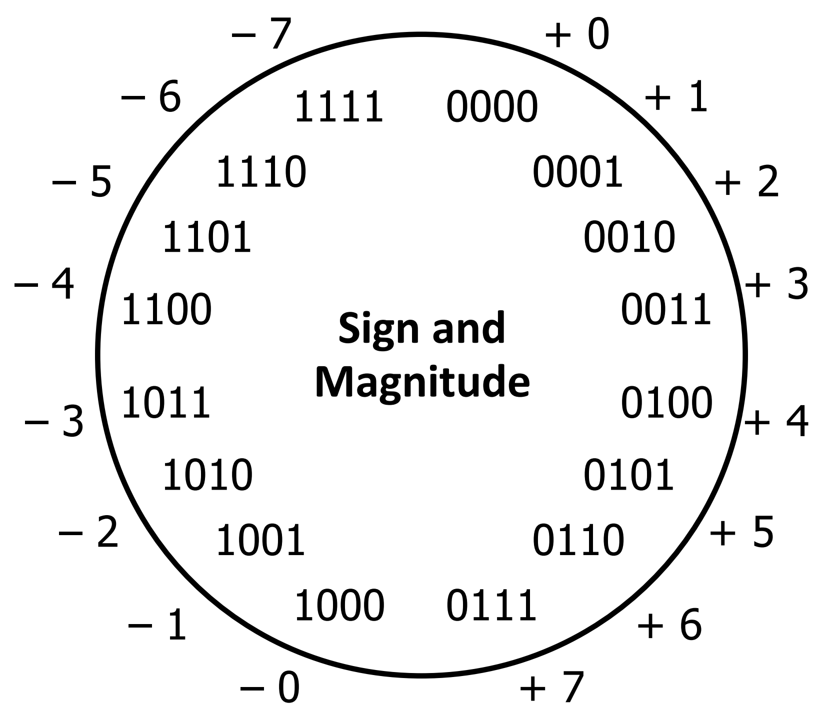
Most Significant Bit

- Designate the high-order bit (MSB) as the "sign bit"
  - `sign=0`: positive numbers; `sign=1`: negative numbers
- Benefits:
  - Using MSB as sign bit matches positive numbers with unsigned
  - All zeros encoding is still = 0
- Examples (8 bits):
  - 0x00 = 00000000_2 is non-negative, because the sign bit is 0
  - 0x7F = 01111111_2 is non-negative ($+127_{10}$)
  - 0x85 = 10000101_2 is negative (-5_{10})
  - 0x80 = 10000000_2 is negative... zero???

Sign and Magnitude

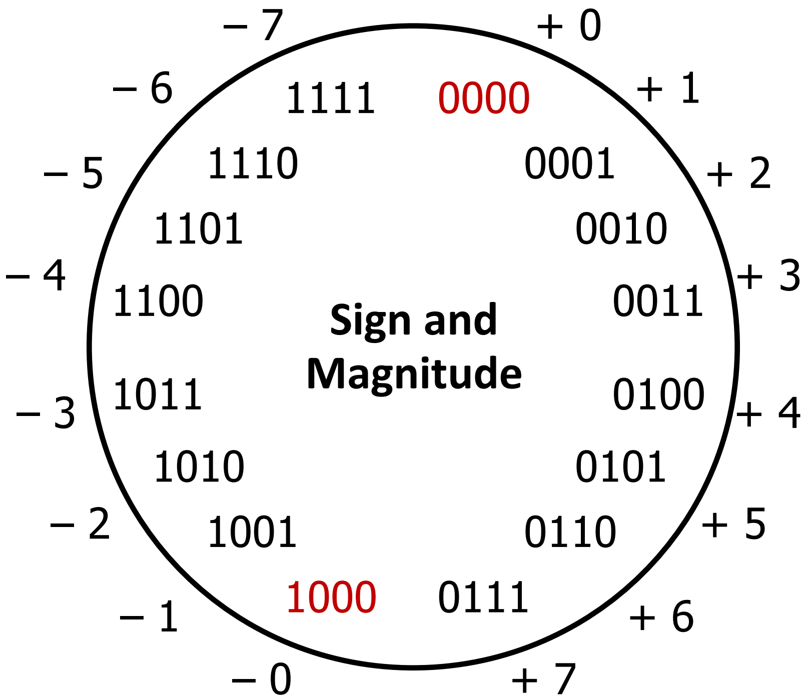
- MSB is the sign bit, rest of the bits are magnitude
- Drawbacks?





Sign and Magnitude

- MSB is the sign bit, rest of the bits are magnitude
- Drawbacks:
  - Two representations of 0 (bad for checking equality)



Sign and Magnitude

- MSB is the sign bit, rest of the bits are magnitude
- Drawbacks:
  - Two representations of 0 (bad for checking equality)
  - Arithmetic is cumbersome
    - Example: `4-3 != 4+(-3)`

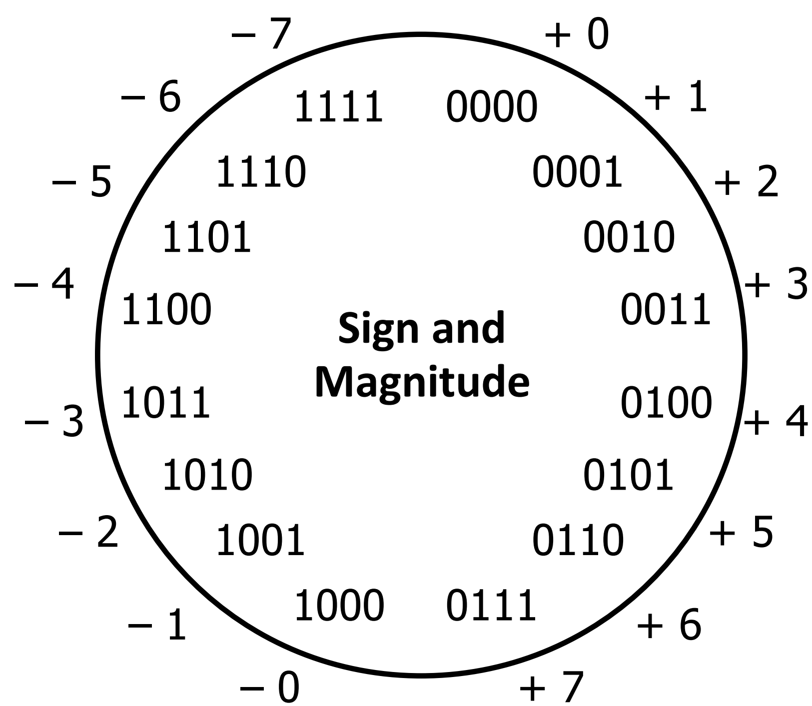
4	0100
– 3	– 0011
1	0001

✓

4	0100
+ -3	+ 1011
-7	1111

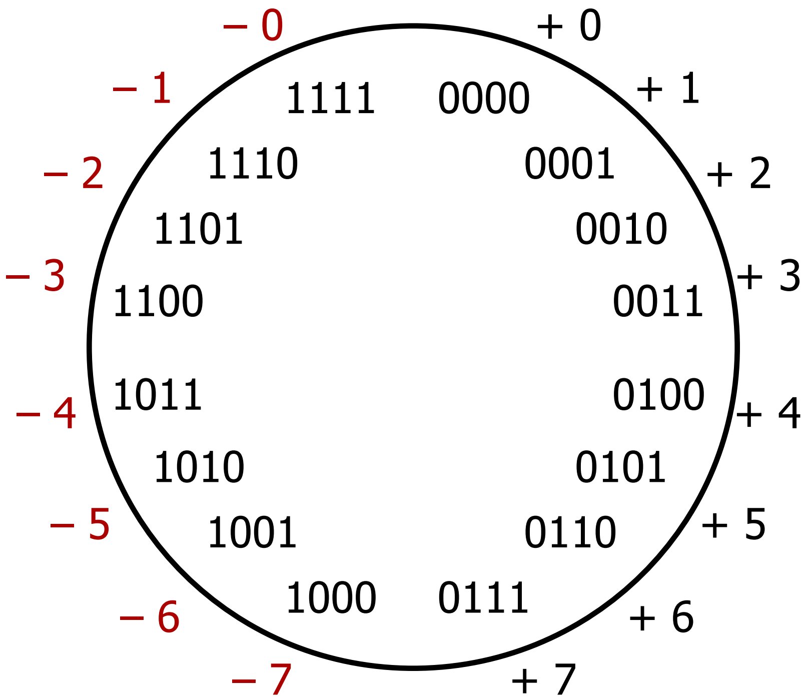
✗

    - Negatives "increment" in wrong direction!



Two's Complement

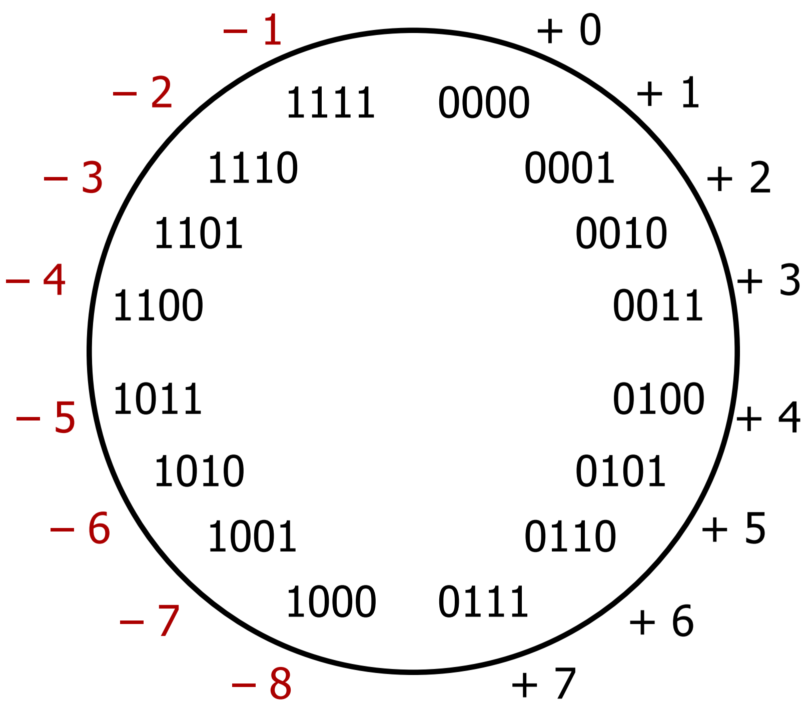
- Let's fix these problems:
  1) "Flip" negative encodings so incrementing works



Two's Complement

- Let's fix these problems:
  1) "Flip" negative encodings so incrementing works
  2) "Shift" negative numbers to eliminate –0

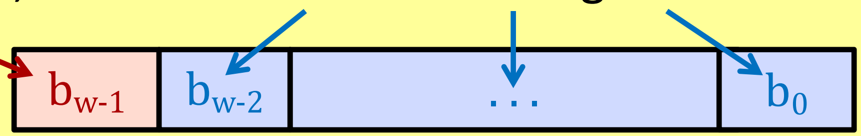
- MSB *still* indicates sign!
  - This is why we represent one more negative than positive number (-2^{N-1} to $2^{N-1}-1$)



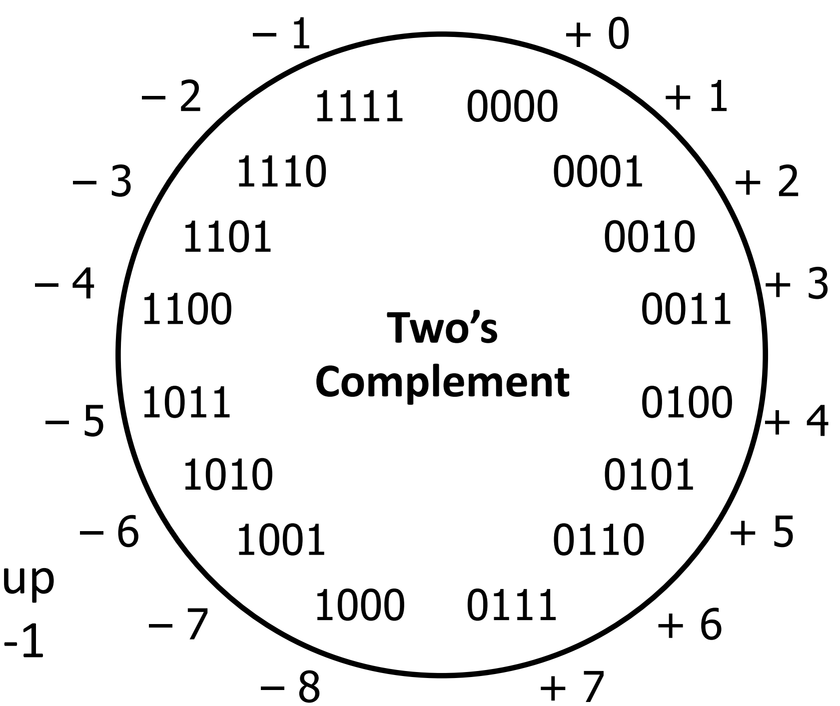
Two's Complement Negatives

- Accomplished with one neat mathematical trick!

$b_{(w-1)}$ has weight $-2^{(w-1)}$, other bits have usual weights $+2^{i}$

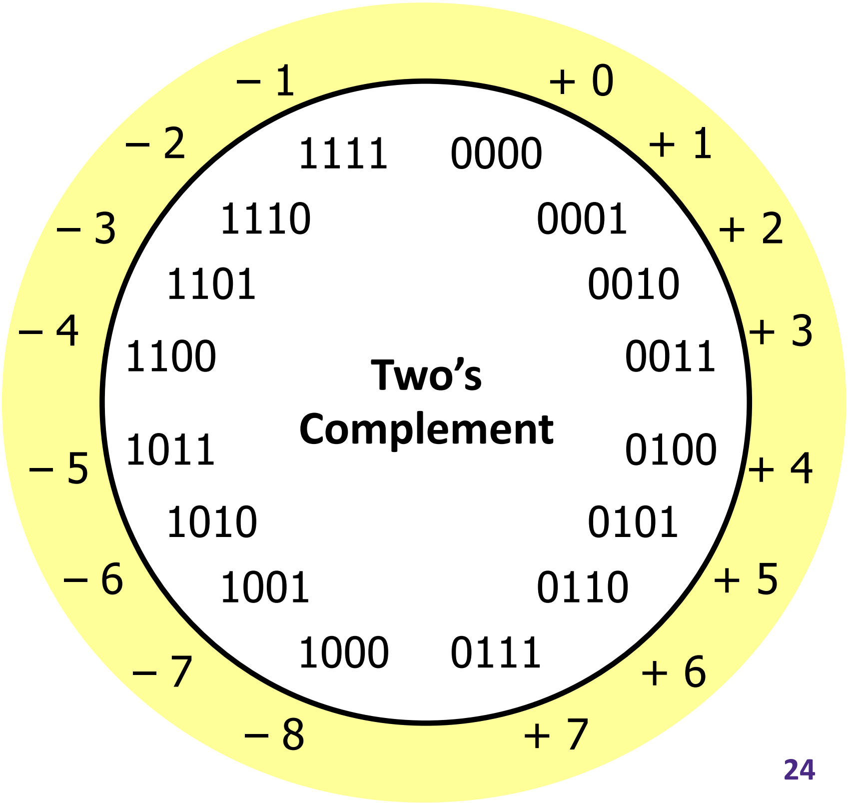


- 4-bit Examples:
  - 1010_2 unsigned:
    $1*2^3+0*2^2+1*2^1+0*2^0 =$ **10**
  - 1010_2 two's complement:
    $-1*2^3+0*2^2+1*2^1+0*2^0 =$ **–6**
- -1 represented as:
  $1111_2 = -2^3+(2^3 – 1)$
  - MSB makes it super negative, add up all the other bits to get back up to -1



Why Two's Complement is So Great

- Roughly same number of (+) and (–) numbers
- Positive number encodings match unsigned
- Simple arithmetic (x + -y = x – y)
- Single zero
- All zeros encoding = 0
- Simple negation procedure:
  - Get negative representation of any integer by taking bitwise complement and then adding one!
    `( ~x + 1 == -x )`



Summary

- Bit-level operators allow for fine-grained manipulations of data
  - Bitwise AND (`&`), OR (`|`), and NOT (`~`) **different than logical** AND (`&&`), OR (`||`), and NOT (`!`)
  - Especially useful with bit masks
- Choice of *encoding scheme* is important
  - Tradeoffs based on size requirements and desired operations
- Integers represented using unsigned and two's complement representations
  - Limited by fixed bit width
  - We'll examine arithmetic operations next lecture