Integers

- **Binary representation of integers**
  - **Unsigned and signed**
  - **Casting in C**
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations

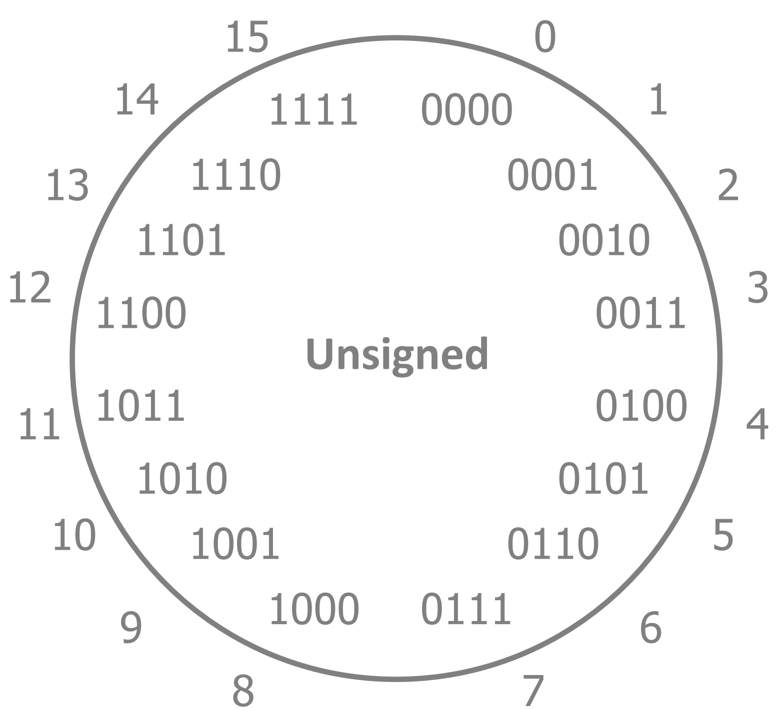
Sign and Magnitude

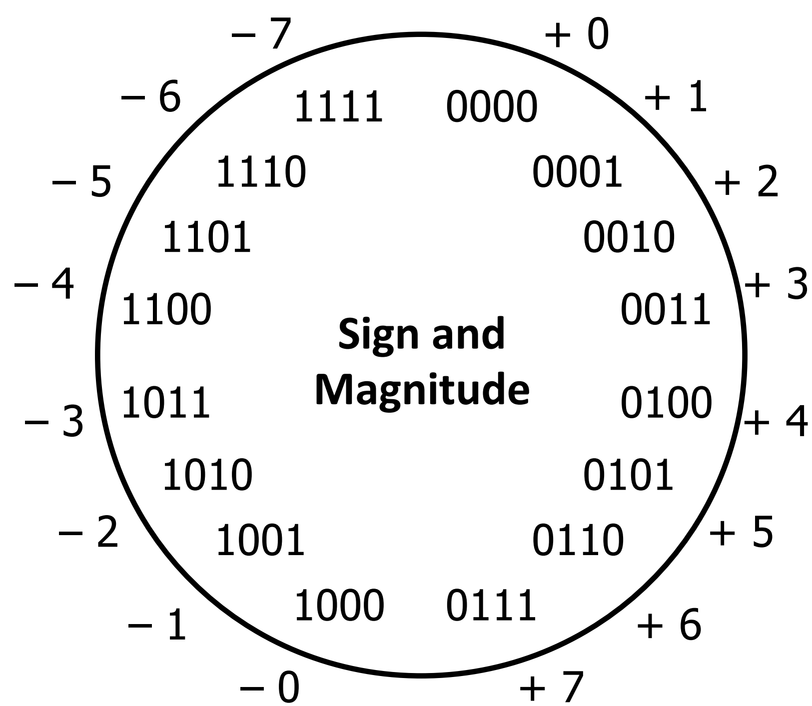
Most Significant Bit

- Designate the high-order bit (MSB) as the "sign bit"
  - `sign=0`: positive numbers; `sign=1`: negative numbers
- Benefits:
  - Using MSB as sign bit matches positive numbers with unsigned
  - All zeros encoding is still = 0
- Examples (8 bits):
  - 0x00 = 00000000_2 is non-negative, because the sign bit is 0
  - 0x7F = 01111111_2 is non-negative ($+127_{10}$)
  - 0x85 = 10000101_2 is negative (-5_{10})
  - 0x80 = 10000000_2 is negative... zero???

Sign and Magnitude

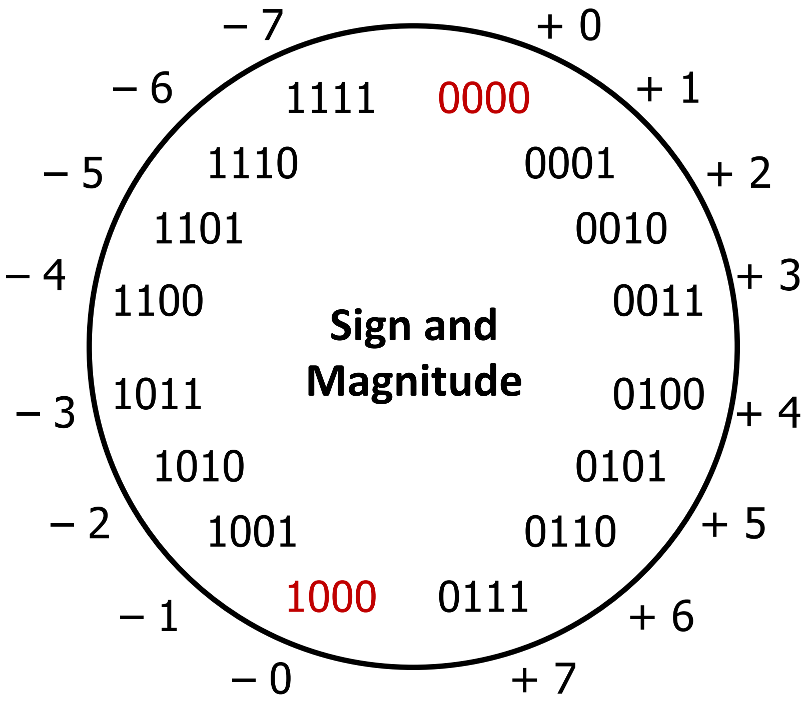
- MSB is the sign bit, rest of the bits are magnitude
- Drawbacks?





Sign and Magnitude

- MSB is the sign bit, rest of the bits are magnitude
- Drawbacks:
  - Two representations of 0 (bad for checking equality)



Sign and Magnitude

- MSB is the sign bit, rest of the bits are magnitude
- Drawbacks:
  - Two representations of 0 (bad for checking equality)
  - Arithmetic is cumbersome
    - Example: `4-3 != 4+(-3)`

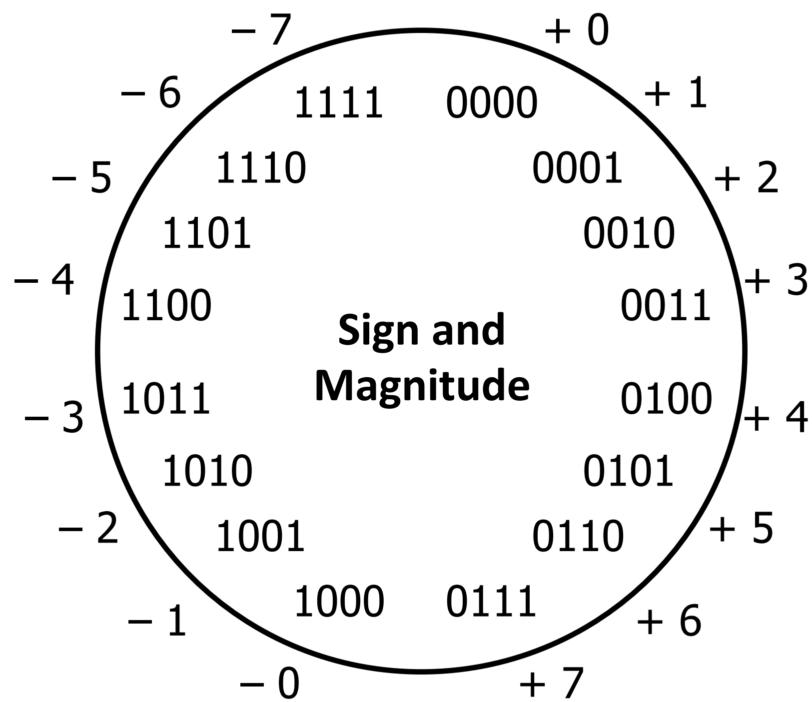
4	0100
– 3	– 0011
1	0001

✓

4	0100
+ -3	+ 1011
-7	1111

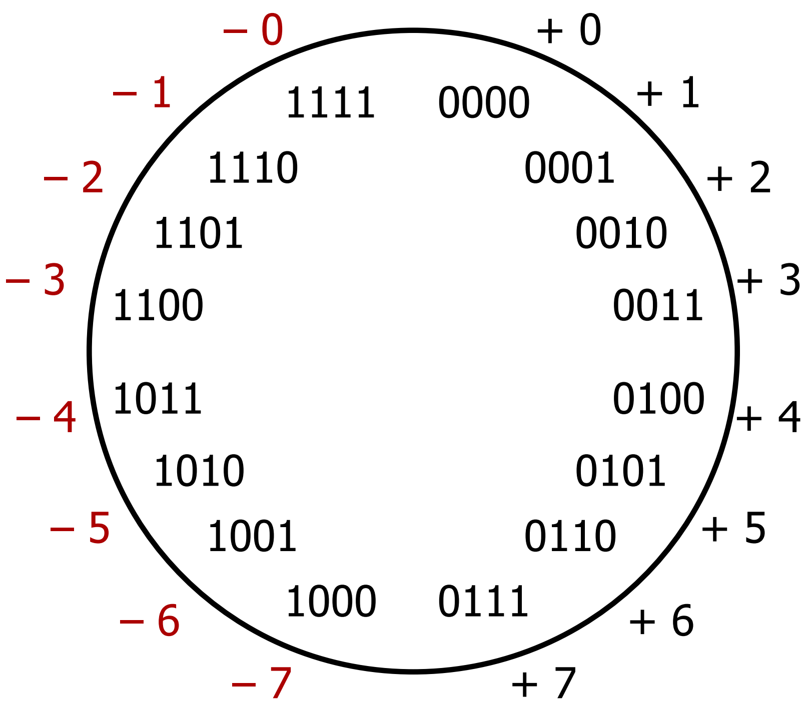
✗

- Negatives "increment" in wrong direction!



Two's Complement

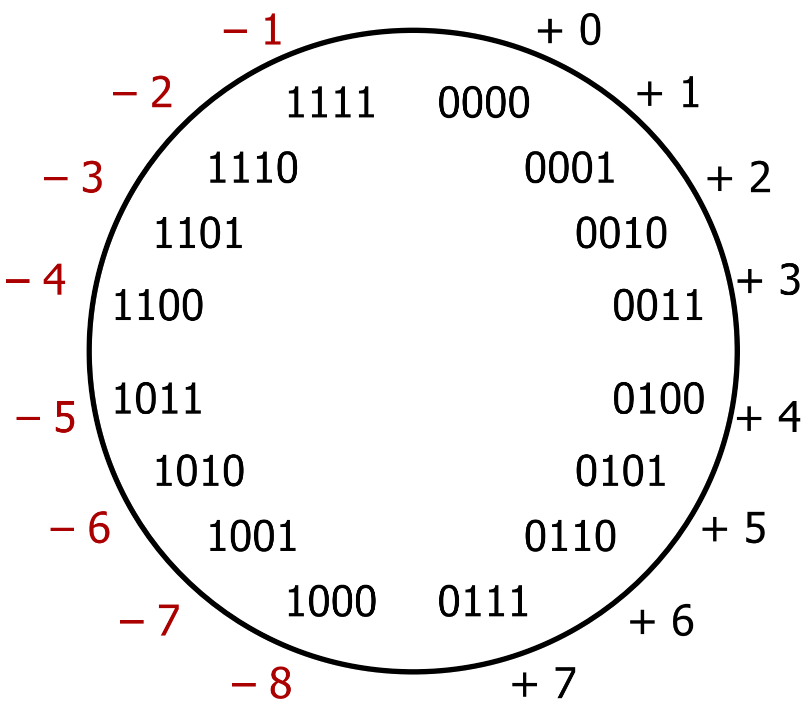
- Let's fix these problems:
  1) "Flip" negative encodings so incrementing works



Two's Complement

- Let's fix these problems:
  1) "Flip" negative encodings so incrementing works
  2) "Shift" negative numbers to eliminate –0

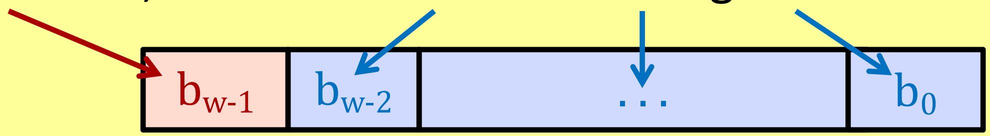
- MSB *still* indicates sign!
  - This is why we represent one more negative than positive number (-2^{N-1} to 2^{N-1} -1)



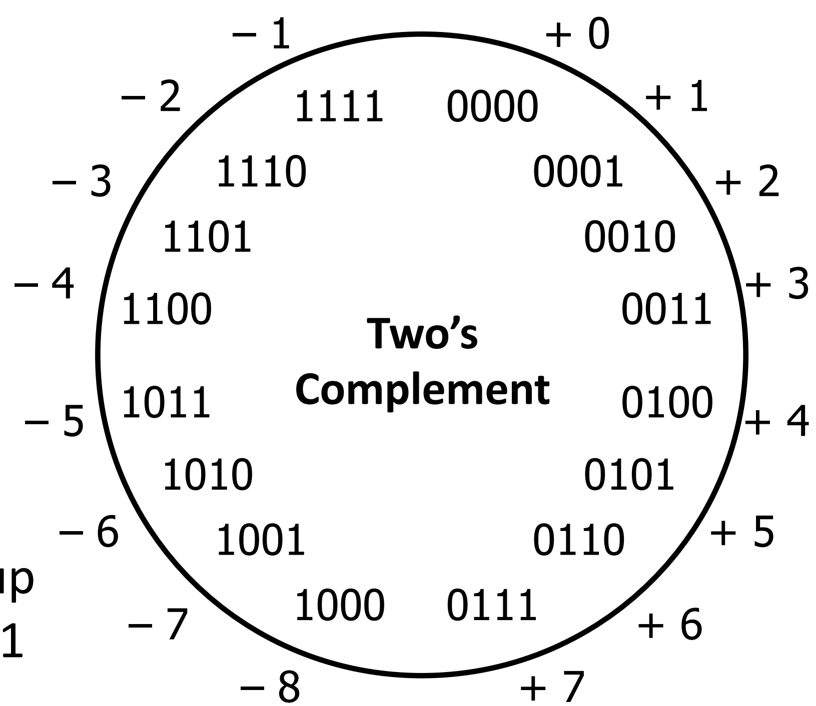
Two’s Complement Negatives

- Accomplished with one neat mathematical trick!

$b_{(w-1)}$ has weight $-2^{(w-1)}$, other bits have usual weights $+2^{i}$

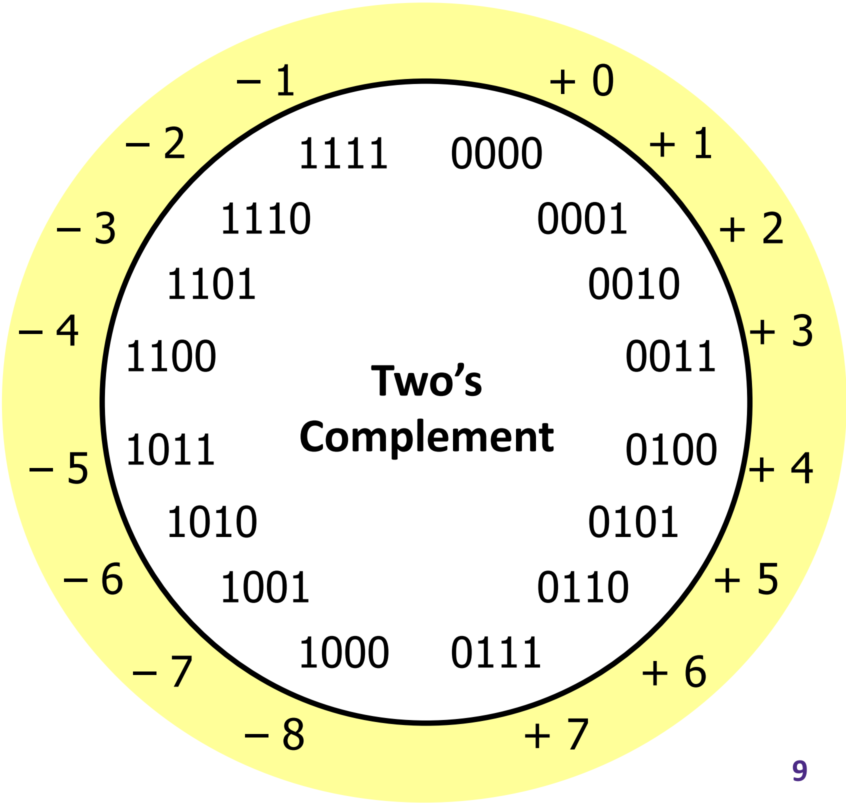


- 4-bit Examples:
  - 1010_2 unsigned:
    $1*2^3+0*2^2+1*2^1+0*2^0$ = **10**
  - 1010_2 two’s complement:
    $-1*2^3+0*2^2+1*2^1+0*2^0$ = **–6**
- -1 represented as:
  $1111_2 = -2^3+(2^3 – 1)$
  - MSB makes it super negative, add up all the other bits to get back up to -1



Why Two's Complement is So Great

- Roughly same number of (+) and (–) numbers
- Positive number encodings match unsigned
- Simple arithmetic (x + -y = x – y)
- Single zero
- All zeros encoding = 0
- Simple negation procedure:
  - Get negative representation of any integer by taking bitwise complement and then adding one!
    `( ~x + 1 == -x )`



Peer Instruction Question

- Take the 4-bit number encoding `x = 0b1011`
- What's it mean as an **unsigned** 4-bit integer?
- What about **signed**?

A. -4
B. -5
C. 11
D. -3
E. We're lost…

Two's Complement Arithmetic

- The same addition procedure works for both unsigned and two's complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum *modulo* 2^w

- **4-bit Examples:**

4 0100 +3 +0011	-4 1100 +3 +0011	4 0100 -3 +1101
=7	=-1	=1

Why Does Two's Complement Work?

- For all integers x, we want:

$$\begin{array}{r} \textit{bit representation of } x \\ + \textit{ bit representation of } -x \\ \hline 0 \end{array}$$ (ignoring the carry-out bit)

  - Start with an observation: what's `x + ~x`?

```
  00000001      00000010      11000011
+ 11111110    + 11111101    + 00111100
----------    ----------    ----------
```

Why Does Two's Complement Work?

- For all integers x, we want:

$$\begin{array}{r} \textit{bit representation of } \; x \\ + \textit{ bit representation of } -x \\ \hline 0 \end{array}$$

(ignoring the carry-out bit)

  - Start with an observation: what's `x + ~x`?

```
   00000001        00000010        11000011
+  11111110     +  11111101     +  00111100
-----------     -----------     -----------
   11111111        11111111        11111111
```

  - So what's `x + ~x + 1`

Why Does Two's Complement Work?

- For all integers x, we want:

$$\begin{array}{r} \textit{bit representation of } \; x \\ + \; \textit{bit representation of} -x \\ \hline 0 \end{array}$$ (ignoring the carry-out bit)

  - Start with an observation: what's `x + ~x`?

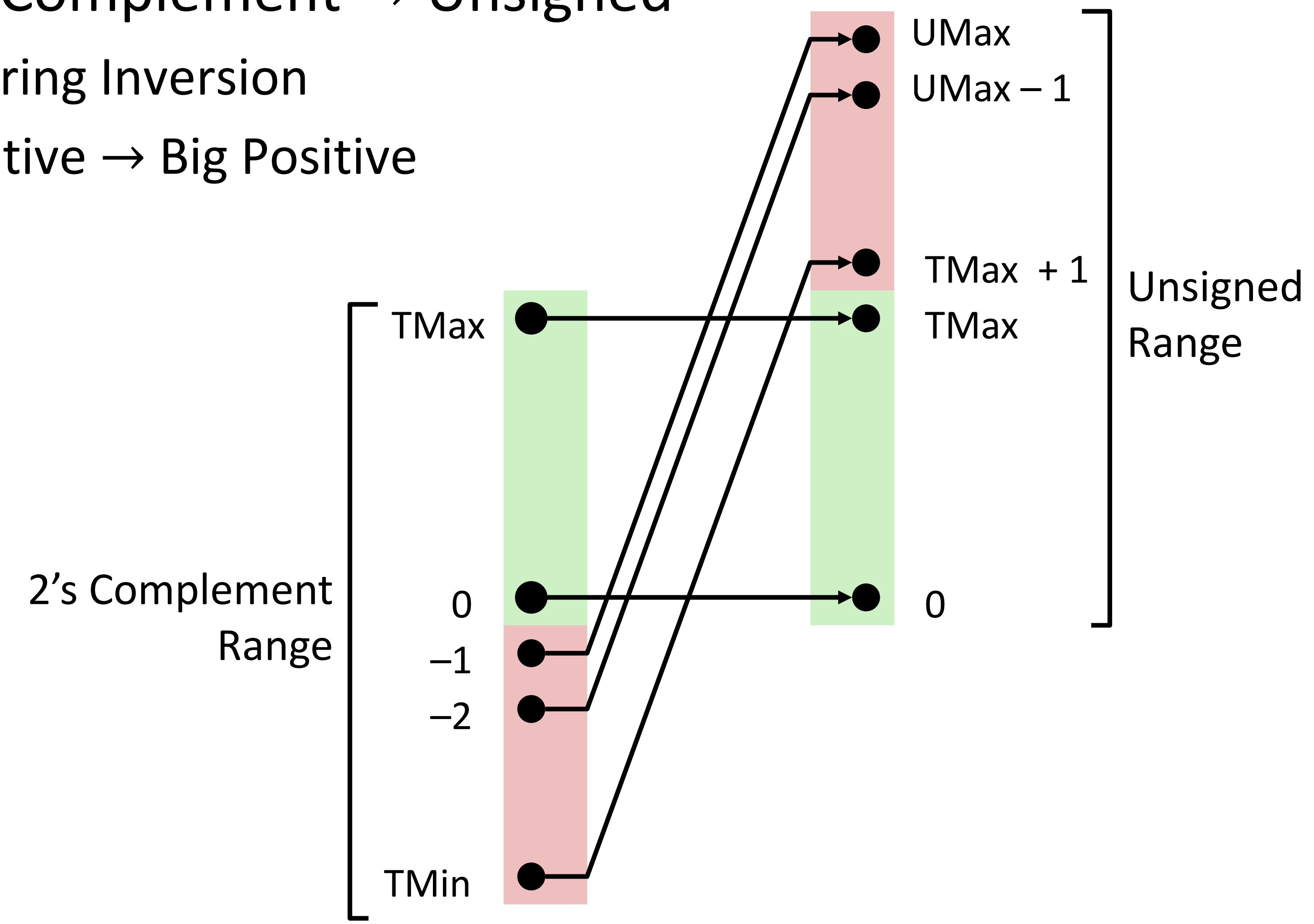
```
  00000001        00000010        11000011
+ 11111110      + 11111101      + 00111100
----------      ----------      ----------
  11111111        11111111        11111111
```

  - So what's `x + ~x + 1` **= 0**

`-x == ~x + 1`

Signed/Unsigned Conversion Visualized

- Two's Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Values To Remember

- Unsigned Values
  - UMin = `0b00…0` = 0
  - UMax = `0b11…1` = $2^w - 1$
- Two's Complement Values
  - TMin = `0b10…0` = $-2^{(w-1)}$
  - TMax = `0b01…1` = $2^{(w-1)} - 1$
  - -1 = `0b11…1`
- **Example:** Values for $w = 64$

	Decimal	Hex
UMax	18,446,744,073,709,551,615	FF FF FF FF FF FF FF FF
TMax	9,223,372,036,854,775,807	7F FF FF FF FF FF FF FF
TMin	-9,223,372,036,854,775,808	80 00 00 00 00 00 00 00
-1	-1	FF FF FF FF FF FF FF FF
0	0	00 00 00 00 00 00 00 00

In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - **int** tx, ty;
    - **unsigned int** ux, uy;
  - *Explicit* casting
    - tx = (**int**) ux;
    - uy = (**unsigned int**) ty;
  - *Implicit* casting can occur during assignments or function calls
    - tx = ux;
    - uy = ty;

Casting Surprises

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use “U” (or “u”) suffix to explicitly force *unsigned*
    - **Examples:** 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then **signed values are implicitly cast to unsigned**
  - Including comparison operators <, >, ==, <=, >=

Casting Surprises

- 32-bit examples:
  - TMin = -2,147,483,648, TMax = 2,147,483,647

Left Constant	Order	Right Constant	Interpretation
0 0000 0000 0000 0000 0000 0000 0000 0000		**0U** 0000 0000 0000 0000 0000 0000 0000 0000	
-1 1111 1111 1111 1111 1111 1111 1111 1111		**0** 0000 0000 0000 0000 0000 0000 0000 0000	
-1 1111 1111 1111 1111 1111 1111 1111 1111		**0U** 0000 0000 0000 0000 0000 0000 0000 0000	
2147483647 0111 1111 1111 1111 1111 1111 1111 1111		**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	
2147483647U 0111 1111 1111 1111 1111 1111 1111 1111		**-2147483648** 1000 0000 0000 0000 0000 0000 0000 0000	
-1 1111 1111 1111 1111 1111 1111 1111 1111		**-2** 1111 1111 1111 1111 1111 1111 1111 1110	
(unsigned) -1 1111 1111 1111 1111 1111 1111 1111 1111		**-2** 1111 1111 1111 1111 1111 1111 1111 1110	
2147483647 0111 1111 1111 1111 1111 1111 1111 1111		**2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	
2147483647 0111 1111 1111 1111 1111 1111 1111 1111		**(int) 2147483648U** 1000 0000 0000 0000 0000 0000 0000 0000	

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- **Consequences of finite width representations**
  - **Overflow, sign extension**
- Shifting and arithmetic operations

Arithmetic Overflow

Bits	Unsigned	Signed
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

- When a calculation produces a result that can’t be represented in the current encoding scheme
  - Integer range limited by fixed width
  - Can occur in both the positive and negative directions
- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication… oops!

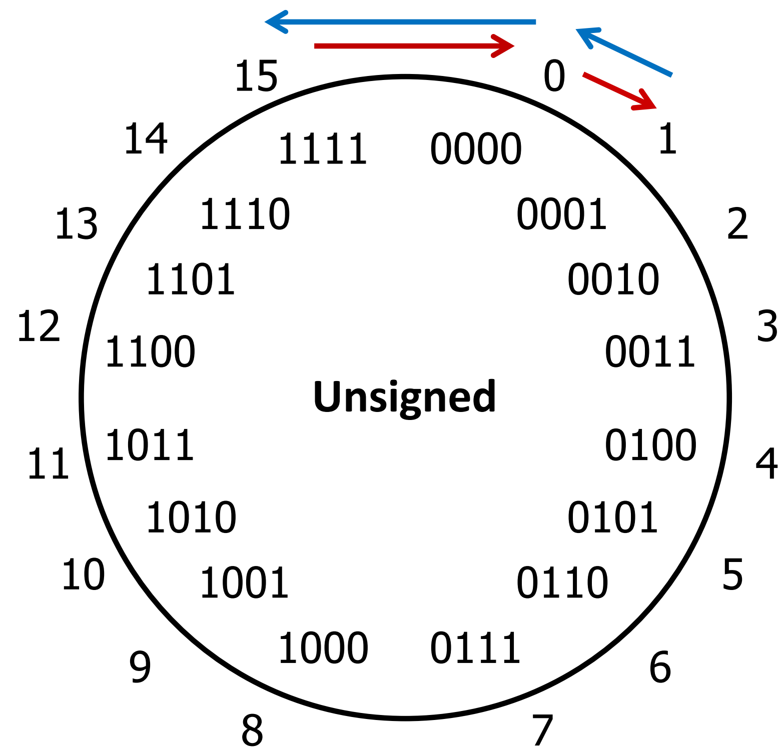
Overflow: Unsigned

- **Addition:** drop carry bit (-2^N)

```
  15        1111
+  2      + 0010
----      ------
  17       10001
   1
```

- **Subtraction:** borrow ($+2^N$)

```
   1       10001
-  2      - 0010
----      ------
  -1        1111
  15
```



$\pm 2^N$ because of modular arithmetic

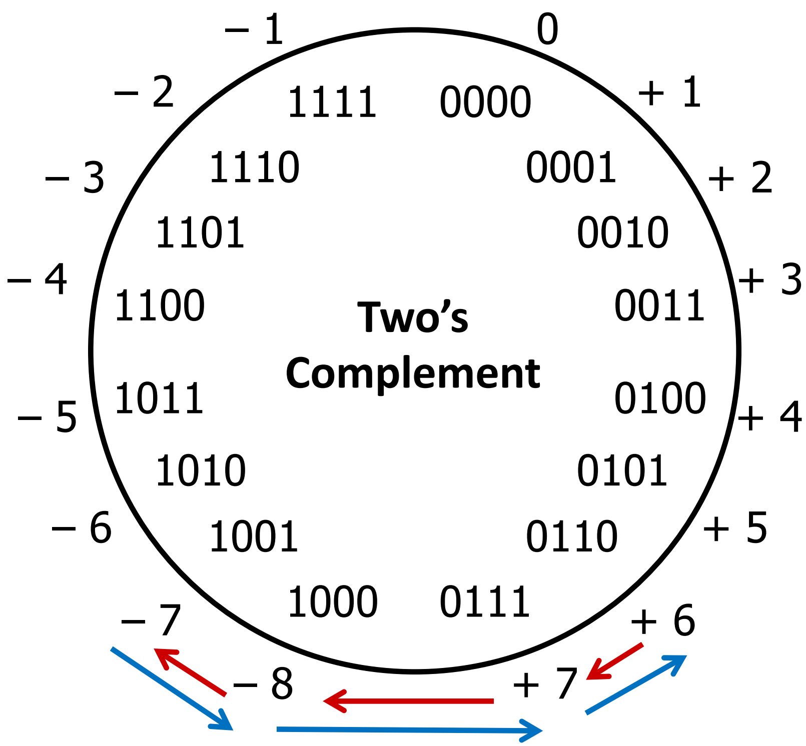
Overflow: Two's Complement

- **Addition:** (+) + (+) = (−) result?

6	0110
+ 3	+ 0011
~~9~~	1001
−7	

- **Subtraction:** (−) + (−) = (+)?

−7	1001
− 3	− 0011
~~−10~~	0110
6	



For signed: overflow if operands have same sign and result's sign is different

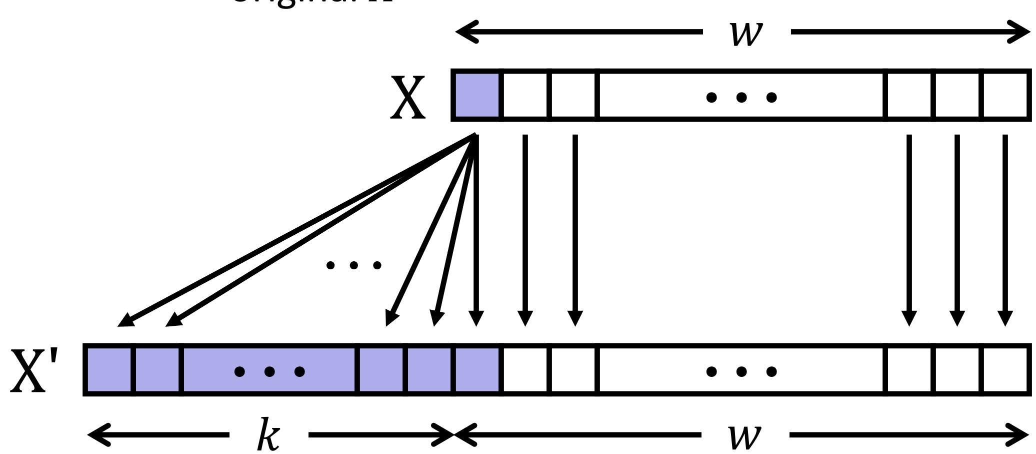
Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* `char` → `short` → `int` → `long`
- **4-bit → 8-bit Example:**
  - Positive Case
    - ✓ Add 0's?
  - Negative Case?

4-bit:	`0010`	=	+2
8-bit:	`00000010`	=	+2

Sign Extension

- **Task:** Given a w-bit signed integer X, convert it to w+k-bit signed integer X' *with the same value*
- **Rule:** Add k copies of sign bit
  - Let x_i be the i-th digit of X in binary
  - $\mathrm{X}' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, \underbrace{x_{w-1}, x_{w-2}, \ldots, x_1, x_0}_{\text{original X}}$



Sign Extension Example

- Convert from smaller to larger integral data types
- C (and Java) automatically performs sign extension when converting to larger types

```
short int x =   12345;
int      ix = (int) x;
short int y =  -12345;
int      iy = (int) y;
```

Var	Decimal	Hex	Binary
x	12345	30 39	00110000 00111001
ix	12345	00 00 30 39	00000000 00000000 00110000 00111001
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

Integers

- Binary representation of integers
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  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- **Shifting and arithmetic operations**

Shift Operations

- Left shift (`x<<n`) bit vector `x` by `n` positions
  - Throw away (drop) extra bits on left
  - Fill with `0`s on right
- Right shift (`x>>n`) bit-vector `x` by `n` positions
  - Throw away (drop) extra bits on right
  - Logical shift (for unsigned values)
    - Fill with `0`s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of `x`

Shift Operations

- Left shift (`x<<n`)
  - Fill with 0s on right
- Right shift (`x>>n`)
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
- Notes:
  - Shifts by `n<0` or `n≥w` (bit width of x) are *undefined*
  - **In C:** behavior of `>>` is determined by compiler
    - depends on data type of `x` (signed/unsigned)
  - **In Java:** logical shift is `>>>` and arithmetic shift is `>>`

	x	0010 0010
	x<<3	0001 0000
logical:	x>>2	0000 1000
arithmetic:	x>>2	0000 1000

	x	1010 0010
	x<<3	0001 0000
logical:	x>>2	0010 1000
arithmetic:	x>>2	1110 1000

Shifting Arithmetic?

- What are the following computing?
  - `x>>n`
    - `0b 0100 >> 1 = 0b 0010`
    - `0b 0100 >> 2 = 0b 0001`
    - Divide by 2^n
  - `x<<n`
    - `0b 0001 << 1 = 0b 0010`
    - `0b 0001 << 2 = 0b 0100`
    - Multiply by 2^n
- Shifting is faster than general multiply and divide operations

Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: $x*2^n$?

			Signed	Unsigned
`x = 25;`	00011001	=	25	25
`L1=x<<2;`	0001100100	=	100	100
`L2=x<<3;`	00011001000	=	-56	200
`L3=x<<4;`	000110010000	=	-112	144

signed overflow

unsigned overflow

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: `x/2`^{n}?

```
xu = 240u;   11110000      = 240
R1u=xu>>3;   00011110000   =  30
R2u=xu>>5;   0000011110000 =   7
```

rounding (down)

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Arithmetic Shift: $x/2^n$?

```
xs = -16;     11110000          = -16
R1s=xu>>3;    11111110000       =  -2
R2s=xu>>5;    1111111110000     =  -1
```

rounding (down)

Peer Instruction Question

For the following expressions, find a value of **`signed char`** `x`, if there exists one, that makes the expression `TRUE`. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:
  - `x == (`**`unsigned char`**`) x`
  - `x >= 128U`
  - `x != (x>>2)<<2`
  - `x == -x`
    - Hint: there are two solutions
  - `(x < 128U) && (x > 0x3F)`

Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just *interpreted* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)
- We can only represent so many numbers in *w* bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding
- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking

BONUS SLIDES

Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1.

- Extract the 2nd most significant byte of an `int`
- Extract the sign bit of a signed `int`
- Conditionals as Boolean expressions

Using Shifts and Masks

- Extract the 2nd most significant *byte* of an `int`:
  - First shift, then mask: `(x>>16) & 0xFF`

x	00000001 00000010 00000011 00000100
x>>16	00000000 00000000 00000001 00000010
0xFF	00000000 00000000 00000000 11111111
(x>>16) & 0xFF	00000000 00000000 00000000 00000010

  - Or first mask, then shift: `(x & 0xFF0000)>>16`

x	00000001 00000010 00000011 00000100
0xFF0000	00000000 11111111 00000000 00000000
x & 0xFF0000	00000000 00000010 00000000 00000000
(x&0xFF0000)>>16	00000000 00000000 00000000 00000010

Using Shifts and Masks

- Extract the *sign bit* of a signed `int`:
  - First shift, then mask: `(x>>31) & 0x1`
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear `1`s possibly shifted in

x	00000001 00000010 00000011 00000100
x>>31	00000000 00000000 00000000 00000000
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000000

x	10000001 00000010 00000011 00000100
x>>31	11111111 11111111 11111111 11111111
0x1	00000000 00000000 00000000 00000001
(x>>31) & 0x1	00000000 00000000 00000000 00000001

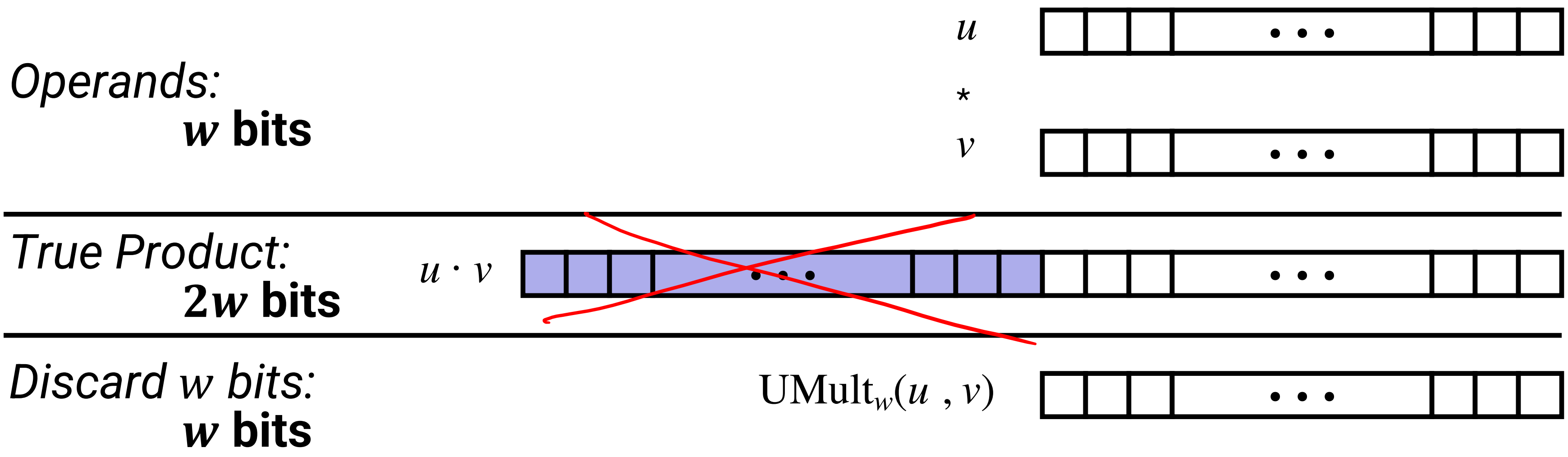
Using Shifts and Masks

- Conditionals as Boolean expressions
  - For **`int`** `x`, what does `(x<<31)>>31` do?

`x=!!123`	`00000000 00000000 00000000 00000001`
`x<<31`	`10000000 00000000 00000000 00000000`
`(x<<31)>>31`	`11111111 11111111 11111111 11111111`
`!x`	`00000000 00000000 00000000 00000000`
`!x<<31`	`00000000 00000000 00000000 00000000`
`(!x<<31)>>31`	`00000000 00000000 00000000 00000000`

  - Can use in place of conditional:
    - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
    - `a=(((x<<31)>>31)&y) | (((!x<<31)>>31)&z);`

Unsigned Multiplication in C



- Standard Multiplication Function
  - Ignores high order *w* bits
- Implements Modular Arithmetic
  - $\text{UMult}_w(u, v) = u \cdot v \mod 2^w$

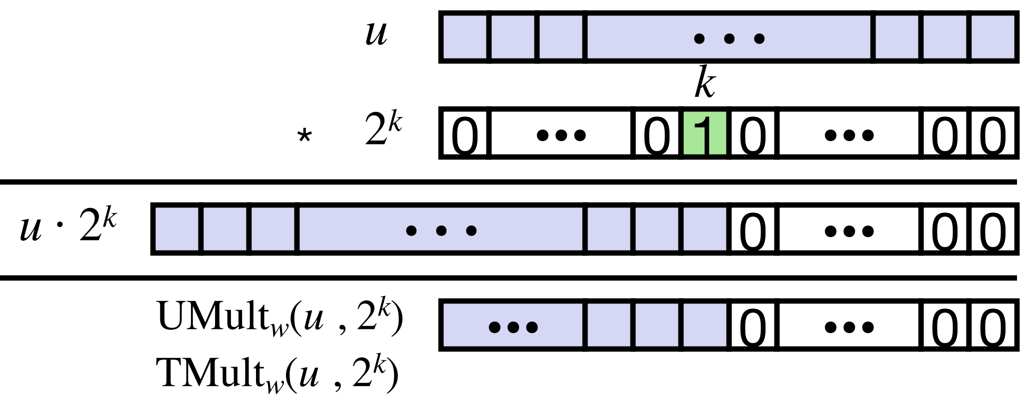
Multiplication with shift and add

- Operation `u<<k` gives `u*2^k`
  - Both signed and unsigned

Operands: ***w*** **bits**

True Product: ***w*** **+** ***k*** **bits**

Discard k bits: ***w*** **bits**



- <u>Examples</u>:
  - `u<<3          ==   u * 8`
  - `u<<5 - u<<3   ==   u * 24`
  - Most machines shift and add faster than multiply
    - ***Compiler generates this code automatically***