CMPT409/815: Advanced Algorithms

Homework Assignment 1

Instructor: Igor Shinkar *Due date: October 7, 2020*

Instructions: Submit your solution to Coursys (if scanned, make sure it's good quality).

Question 1 (20 points) *Consider a random variable X taking values in the positive integers.*

(a) Suppose that $\Pr[X = 1] = 0.1$ *and* $\Pr[X = 2] = 0.2$*. Prove that* $\mathbb{E}[X] \geq 2$.

(b) Suppose that $\Pr[X = 1] = 0.1$ *and* $\Pr[X = 2] = 0.2$*. Prove that for all integers* $t \geq 3$ *it holds that* $\Pr[X \geq t] \leq \frac{\mathbb{E}[X]-0.5}{t}$.

(c) Suppose that $\Pr[X \leq 2\mathbb{E}[X]] = 1$*. Prove* $\Pr[X \leq \frac{\mathbb{E}[X]}{2}] \leq 2/3$.

Question 2 (20 points)

(a) Prove that any graph G on n vertices has at most $n(n-1)/2$ cuts of minimum size.

(b) Prove that the upper bound of $n(n-1)/2$ on the number min cuts is tight.

Hint: (a) Recall Karger's algorithm we saw in class. (b) Consider the cycle graph on n vertices.

Question 3 (20 points) *Denote by $\alpha(G)$ the size of the maximum independent set in G.*

Design an algorithm that given a graph G that contains an independent set of size $\alpha(G) \geq (\frac{1}{2}+\delta)|V|$, outputs an independent set of size $\geq 2\delta|V|$.

Hint: Prove that G contains a vertex cover of size k if and only if it contains an independent set of size $|V| - k$.

Question 4 (20 points) *An r-uniform hypergraph $H = (V, E)$ is a collection of vertices V, and a collection of hyperedges E, where each hyperedge $e \in E$ is a subset of V of size $|e| = r$. The case of $r = 2$ corresponds to graphs.*

Consider the Minimum Vertex Cover *problem for r-uniform hypergraphs: Given an r-uniform hypergraph the goal is to find a collection of vertices $C \subseteq V$ of minimum size such that for every $e \in E$ it holds that $e \cap C \neq \emptyset$. Design a r-approximation algorithm for the problem of finding a minimum vertex cover in an r-uniform hypergraph.*

Question 5 (20 points) *Design a* randomized *algorithm that gets a graph $G = (V, E)$ and outputs (with high probability) a partition of $V = V_1 \cup V_2$ such that at most $|E|/2$ edges have their endpoints in the same V_i.*