CMPT409/815: Advanced Algorithms

Homework Assignment 2

Instructor: Igor Shinkar *Due date: October 21, 2020*

Instructions: Submit your solution to Coursys (if scanned, make sure it is good quality).

Question 1 (20 points) *Design an polynomial time algorithm for the Set-Cover problem with the following guarantee. The input is a universe U of size n, sets $S_1, S_2, \ldots, S_m \subseteq U$, and a parameter k such that there are k sets that cover U. The algorithm outputs a collection of t sets that cover U such that $t \leq k + k\ln(n/k)$.*

Question 2 (25 points)

(a) Design a polynomial time algorithm that gets as input a graph G on n vertices, that is guaranteed to have a clique of size $n/log^3(n)$. The output of the algorithm is a clique in G of size $\log(n)/\log\log(n)$.

(b) Does the previous item imply that Max-Clique admits a $O(\log^3(n)/n)$-approximation algorithm?

Question 3 (20 points) *Design a randomized algorithm that gets a bipartite graph $G = (V, E)$, and with high probability outputs a perfect matching in G. You may use as a subroutine the algorithm we saw in class for the decision version of this problem.*

Question 4 (25 points) *Design a polynomial time algorithm that gets as input a general* (not necessarily bipartite) *graph $G = (V, E)$, and with high probability decides correctly whether G contains a perfect matching.*

Instructions: Consider the $|V \times |V|$ matrix Z defined as

$$Z(i,j) = \begin{cases} X_{i,j} & \text{if } (i,j) \in E \text{ and } i < j \\ -X_{j,i} & \text{if } (i,j) \in E \text{ and } j < i \\ 0 & \text{otherwise} \end{cases},$$

where $X_{i,j}$ are formal variables. Prove that $det(Z)$ is identically zero if and only if G does not contain a perfect matching.

Question 5 (25 points) *In the* UNIQUE-CLIQUE *problem the input is a graph $G = (V, E)$ and an integer $k > 0$. An algorithm is said to solve the* UNIQUE-CLIQUE *problem if it satisfies the following guarantees.*

YES case *: If G has a clique of size k and the maximum size clique is unique, the algorithm outputs YES.*

NO case *: If G has no clique of size k, the algorithm must output NO.*

Remark*: If G has a clique of size at least k, and has more than one clique of maximum size, the algorithm may output anything.*

Prove that if UNIQUE-CLIQUE *can be solved in polynomial time, then the* MAX-CLIQUE *problem can be solved using a poly-time randomized algorithm.*

In order to do it, show a randomized reduction from the MAX-CLIQUE *problem to the* UNIQUE-CLIQUE *problem. That is, show a randomized polynomial time reduction (an algorithm) that gets a graph H and a parameter $k' > 0$, and outputs a graph G and a parameter k and satisfies the following guarantees.*

YES case : *If H has a clique of size at least k', then*

1. *G has a clique of size at least k, and*
2. $\Pr[$*G has a* unique *clique of maximum size*$] > 0.9$.

NO case : *If H has no clique of size k', then G has no clique of size k.*

(Hint: Use the isolation lemma.)