CMPT409/815: Advanced Algorithms

Homework Assignment 3

Instructor: Igor Shinkar *Due date: November 11, 2020*

Instructions: Submit your solution to Coursys (if scanned, make sure it is good quality).

Question 1 (20 points) *Solve the following linear problem.*

$$\text{minimize}_{x,y,z,w\in\mathbb{R}} \quad 3x+5y+2z+w$$

$$\text{subject to} \begin{cases} x+3y+2w & \geq 1 \\ x+3y+z & \geq 2 \\ 2x+6y+z-w & \geq 2 \\ x,y,z,w \geq 0 \end{cases}$$

Question 2 (25 points)

(a) **[10 points]** *Write an integer linear programming (ILP) formulation for the 3-coloring problem. (There is more than one way to write such an ILP. You may choose any formulation)*

(b) **[5 points]** *Relax the ILP to LP, and show a graph that is not 3-colorable that has a feasible solution to your LP. (Note that this is a feasibility problem, and not an optimization problem)*

(c) **[10 points]** *Write the Dual LP for the linear program in the previous item. For this you will need to add an objective function to the primal LP.*

Question 3 (25 points) *In the MAX-3-CNF problem the input is a 3-CNF formula Φ, and the goal is to find an assignment that that maximizes the number of satisfied clauses.*

(a) **[10 points]** *Write a CSP formulation for the MAX-3-CNF problem.*

(b) **[15 points]** *Write the level-3 Sherali-Adams LP for the MAX-3-CNF problem. Specify explicitly the variables, the constraints, and the objective function.*

Question 4 (30 points) *In the maximum directed cut problem (MAX-DICUT), the input is a directed graph $G=(V,E)$, and for each directed edge $(i\to j)\in E$ we have a non-negative weight $w_{(i\to j)}\in\mathbb{R}^+$.*

The goal is to partition V into two sets (U,W) such that and $W=V\setminus U$ so as to maximize the total weight of the arcs going from U to W (that is, $\sum_{\substack{(i\to j)\in E \\ i\in U, j\in W}} w_{i,j}$).

(a) **[10 points]** *Give a polynomial time randomized 1/4-approximation algorithm for this problem. That is the randomized algorithm returns a partition $V=(U,W=V\setminus U)$ such that the expected total weight of the arcs going from U to W is at least 1/4 of the optimum.*

(b) **[5 points]** *Show that the following LP is a relaxation of the MAX-DICUT problem:*

$$\text{maximize} \sum_{(i \to j) \in E} w_{i,j} z_{i,j}$$

$$\text{subject to} \begin{cases} z_{(i \to j)} \leq x_i & \forall (i \to j) \in E \\ z_{(i \to j)} \leq 1 - x_j & \forall (i \to j) \in E \\ 0 \leq z_{(i \to j)} \leq 1 & \forall (i \to j) \in E \\ 0 \leq x_i \leq 1 & \forall v \in V \end{cases}$$

Explain what the variables $z_{i,j}$ and x_i correspond to in an integral solution.

(c) **[15 points]** *Consider the randomized rounding algorithm for MAX-DICUT that solves the LP above and puts each vertex i in U independently with probability $\frac{1}{4} + \frac{x_i}{2}$. Show that this gives a randomized 1/2-approximation algorithm for MAX-DICUT, i.e., in the output the expected total weight of the arcs going from U to W is at least $1/2$ of the optimum.*

(Hint: Given a solution to LP relate the probability that edge $(i \to j)$ is in the cut (U, W) to the LP value of $z_{i,j}$.)