CMPT409/815: Advanced Algorithms

Homework Assignment 4

Instructor: Igor Shinkar *Due date: December 2, 2020*

Instructions: Submit your solution to Coursys (if scanned, make sure it is good quality).

Question 1 (25 points) *Given a graph $G = (V, E)$ consider the SDP for MAX-CUT we saw in class.*

$$\text{maximize} \sum_{(u,v)\in E)} \frac{1}{2}\langle 1 - \langle x_v, x_u\rangle\rangle$$

$$\text{subject to}: \quad \langle x_v, x_v\rangle = 1 \quad \textit{for all } v \in V.$$

Prove that the integrality gap of this SDP is bounded away from 1. That is, prove that there is a graph G and some absolute constant $c > 1$ such that $OPT(SDP) > c \cdot \max - \text{cut}(G)$.

(Hint: Try proving it on the complete graph on 3 vertices, $G = K_3$.)

Question 2 (25 points) *Given a graph $G = (V, E)$ consider the following SDP for MAX-CUT.*

$$\text{maximize} \quad \sum_{(u,v)\in E)} \frac{1}{2}\langle 1 - \langle x_v, x_u\rangle\rangle$$

$$\begin{aligned} \text{subject to}: \quad & \langle x_v, x_v\rangle = 1 \quad \textit{for all } v \in V \\ & \langle x_u, x_v\rangle + \langle x_v, x_w\rangle + \langle x_u, x_w\rangle \geq -2 \quad \textit{for all } (u,v),(v,w),(u,w) \in E \end{aligned}$$

Prove that the SDP is a relaxation of the max-cut problem.

Question 3 (30 points)

(a) Use the algorithms for graph coloring we saw in class to design an algorithm that colors a 3-colorable graph with $\tilde{O}(n^{1/4})$ colors.

(b) Use the previous item to design a polynomial time algorithm that gets a 4-colorable graph G and outputs a legal $\tilde{O}(n^{4/7})$ coloring of G.

Question 4 (20 points) *Given an n-vertex graph $G = (V, E)$ consider the following SDP.*

$$\begin{aligned} \textit{find a feasible solution:} \quad & v_1, \ldots, v_n \in \mathbb{R}^n \\ & \langle v_i, v_j\rangle = -\frac{1}{k-1} \quad \forall (i,j) \in E \\ & \|v_i\| = 1 \end{aligned}$$

Prove that if G is k-colorable, then the SDP has a feasible solution.

(Hint: prove it for $k = 4$ first)